Data-based linear systems and control theory

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Dedicated to:

Tessa by Henk

Berfu and *Luna* by Kanat

Ellen, *Renée*, *Lucas* and *Jostein* by Harry

Preface

Origin of the book

This book originates from research mainly performed at the Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence at the University of Groningen in the period between 2019 and 2025. The goal of this research has been to establish a unifying framework for a data-based analysis and control theory. The idea is to analyze system properties and to design controllers for dynamical systems, directly from data and without the use of explicit mathematical models. This is a radical departure from mainstream systems and control theory where mathematical models are the central objects of study. This book aims at rewriting parts of systems and control theory by placing time series data at the forefront. There are several motivations for this work. Due to technological advancements, modern engineering systems are so complex that obtaining models from first principles is not feasible. On the other hand, these systems produce massive amounts of data that can be readily harvested. Given the absence of mathematical models, the question of how to utilize the data for analysis and control design is therefore highly relevant. Overall, the development of this book aligns with a general trend in science and engineering towards the extensive use of data, as witnessed by the artificial intelligence boom.

Contents of the book

This book intends to provide a comprehensive framework for data-driven system analysis, control design and modeling. We will focus on discrete-time linear time-invariant systems and data that can be either noisy or noise-free. The central concept used within the book is the notion of *data informativity*. Among other things, the data informativity framework enables the design of controllers from data that do not necessarily satisfy restrictive requirements like persistent excitation. The book begins with a historical perspective in Chapter 1, which is followed by an introduction to the data informativity framework in Chapter 2.

The main body of the book contains five parts. In Part I, we apply the informativity framework to deal with a range of data-based system analysis problems. These problems include deciding on the basis of data whether a system is, for example, controllable, stabilizable or stable. Part II focuses on data-driven control design. Here, we study problems like designing stabilizing and optimal controllers on the basis of data. In Part III, we apply the data informativity framework to the problem of system identification. First, we will provide conditions on the data under which the data-generating system can be uniquely identified. Subsequently, we exploit these conditions to develop online experiment design methods. Part IV studies reduced order modeling on the basis of measured data. This problem is approached from two different angles, namely balanced truncation and moment matching. Finally, Part V is an appendix and is devoted to notational conventions, along with a detailed discussion of the quadratic matrix inequalities used throughout the book. We recommend reading the basic notation in Section A.1 before reading the chapters in the main body of this book.

Intended audience and teaching instructions

The intended audience of the book includes researchers who want to deepen their understanding of data-driven control, and practitioners and engineers interested in applying data-driven control techniques. The book is also suitable for master and PhD students in engineering and mathematics programmes.

To use the book as lecture notes for graduate level courses, a selection of topics can be made. For instance, Chapter 1–2 and a selection of topics from Parts I and II of this book have been used in a master course on data-based analysis and control at the University of Groningen. These topics can be covered in around 30 hours of lecture time.

The required background for the book is a good command of linear algebra and calculus, and basic linear systems theory. The book aims at a self-contained treatment of data-driven systems and control, also including a chapter on mathematical results and their accompanying proofs.

Acknowledgements

We thank all of our collaborators and co-authors who contributed to the papers on which this book is based. We would especially like to thank Jaap Eising for all the valuable discussions and the great collaboration leading to the first papers on data informativity. His comments on some of the earlier versions of chapters were very helpful to improve the quality of this book. We want to take the opportunity to thank the many colleagues who have commented on and provided constructive remarks on the draft version of the book. Finally, we would like to thank the master and PhD students in Groningen who have worked on topics related to data-driven control.

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1

Introduction and historical perspective

In this chapter we give a general introduction to the topic of data-driven control by means of a literature overview. In particular, we will follow a timeline highlighting the main developments in data-driven control. We will also delve into the details of some methods that are most relevant for this book, including the fundamental lemma by Willems et al., and its applications like subspace identification, data-driven simulation and tracking, data-enabled predictive control, and data-driven stabilization. These examples serve as a motivation for the material developed in the rest of the book.

1.1 Introduction

In broad terms, systems and control theory deals with the problem of making a concrete physical system behave according to certain desired specifications. In order to achieve this desired behavior, the system can be interconnected with a device, called a controller. The problem of finding a mathematical description of such a controller is called the *control design problem*.

To obtain a mathematical description of a controller for a to-be-controlled physical system, a possible first step is to obtain a mathematical model of the physical system. Such a mathematical model can take many forms. For example, the model could be in terms of ordinary or partial differential equations, difference equations, or transfer matrices.

There are several ways to obtain a mathematical model for the physical system. The usual way is to apply the basic physical laws that are satisfied by the variables appearing in the system. This method is called *first principles modeling*. For example, for electro-mechanical systems, the set of basic physical laws that govern the behavior of the variables in the system (conservation laws, Newton's laws, Kirchhoff's laws, etc.) form a mathematical model.

An alternative way to obtain a model is to do experiments on the physical system: certain external variables in the physical system are set to take particular values, while at the same time other variables are measured. In this way, one obtains *data* on the system that can be used to find mathematical descriptions of laws that are obeyed by the system variables, thus obtaining a model. This method is called *system identification*.

The second step in a control design problem is to decide which desired behavior we would like the physical system to have. Very often, this desired behavior can be formalized by requiring the mathematical model to have certain qualitative or quantitative mathematical properties. Together, these properties form the *design objective*.

Based on the mathematical model of the physical system and the design objective, the third, ultimate, step is to design a mathematical model of a suitable controller. This approach, leading from a model and a design objective (or list of design specifications) to a controller is an important paradigm in systems and control, and is often called *model-based control*. Indeed, many existing control design techniques rely on a system model, represented by, for example, a state-space system or transfer matrix.

In this book, we will deal with an approach to control design that circumvents the step of finding a mathematical model of the to-be-controlled system. This alternative approach deals with the problem of synthesizing control laws directly on the basis of measured data, and is called the *data-driven approach* to control design. Of course, one can argue that also the combination of system identification and model-based control as described above is an instance of data-driven control design. Indeed, methods using this combination are often called *indirect* methods of data-driven control, consisting of the two-step process of data-driven modeling (i.e., system identification) followed by model-based control.

In addition to the above indirect methods, we distinguish *direct* methods to data-driven control design. These direct approaches focus on directly mapping data to controllers without an intermediate step of system identification. Both paradigms have different pros and cons. For example, identification might be expensive and the obtained model may not always be useful for the intended control design problem. In addition, in many situations unique system identification is impossible, for example because the data are corrupted by noise and do not contain sufficient information about the underlying system. In contrast, direct data-driven control design has the premise of being an end-to-end approach, requiring less expert knowledge. It could therefore be the preferred choice for practitioners. However, in comparison to the maturity of system identification, the theory of direct data-driven control is still very much under development. In fact, the early 2020s witnessed a surge of research activity in direct data-driven control. Some of these results have been summarized in survey papers, such as those in the Control Systems Magazine double special issue on data-driven control [148, 149].

With the current book, we aim at giving a general treatment of direct data-driven analysis and control design from the unifying perspective of *data informativity*. The overarching question that will be studied is how to use only the data obtained from the unknown system to verify its system-theoretic properties and to construct controllers. Interestingly, the data informativity framework does not only shed light on direct data-driven analysis and control, but also has consequences for modeling. This will be demonstrated by studying system identification and reduced order data-driven modeling through the lens of data informativity.

1.2 Historical perspective

The purpose of this section is to provide a bird's-eye view of the developments of data-driven control in the period between 1950 and 2025, following the coarse timeline in Figure 1.1. We emphasize that this timeline is by no means exhaustive, but it contains many of the main contributions. The high-level discussion will be followed by five detailed subsections summarizing results that are most closely related to this book, as well as an overview of further developments.

Early developments in data-driven control mainly include the combination of system identification (i.e., data-driven modeling) [59], followed by control design based on the identified model. We mention contributions to *prediction error methods* [95, 96] in the 1970s and 1980s, and *subspace identification* [115, 165, 177, 187] in the late 1980s and 1990s. The analysis and control design methods developed in this book depart from identification-based approaches, in the sense that the intermediate modeling step is skipped. Instead, in this book we design controllers and analyze system properties directly using time series data. As we will see, this direct data-driven control approach is powerful especially in situations when the data do not enable unique system identification [175]. Although the direct approach is a radical departure from the indirect one, we do mention some important parallels with the system identification literature. First of all, one of the main ingredients of the *data informativity framework*, used throughout this book, is the set of all data-consistent systems. This is in line with *set membership identification* (SMI) methods [92, 110], where sets of data-consistent systems also play an important role. In SMI, these sets are typically called feasible system sets. Through the lens of SMI, the main contributions of this book are to provide easily verifiable conditions on the data under which all systems in the feasible system set have a certain system-theoretic property, and when all of these systems can be controlled by a single controller. The material of this book also resonates well with the development of *identification for control* [58, 62, 164] in the late 1980s and 1990s. The main idea of this movement was to take into account the eventual purpose of the model during the identification stage. In this way, the identified model is suitable for its intended application, which is typically control design. In this book we take one step further: by eliminating the need for intermediate system identification, we naturally place the intended control design task at the forefront.

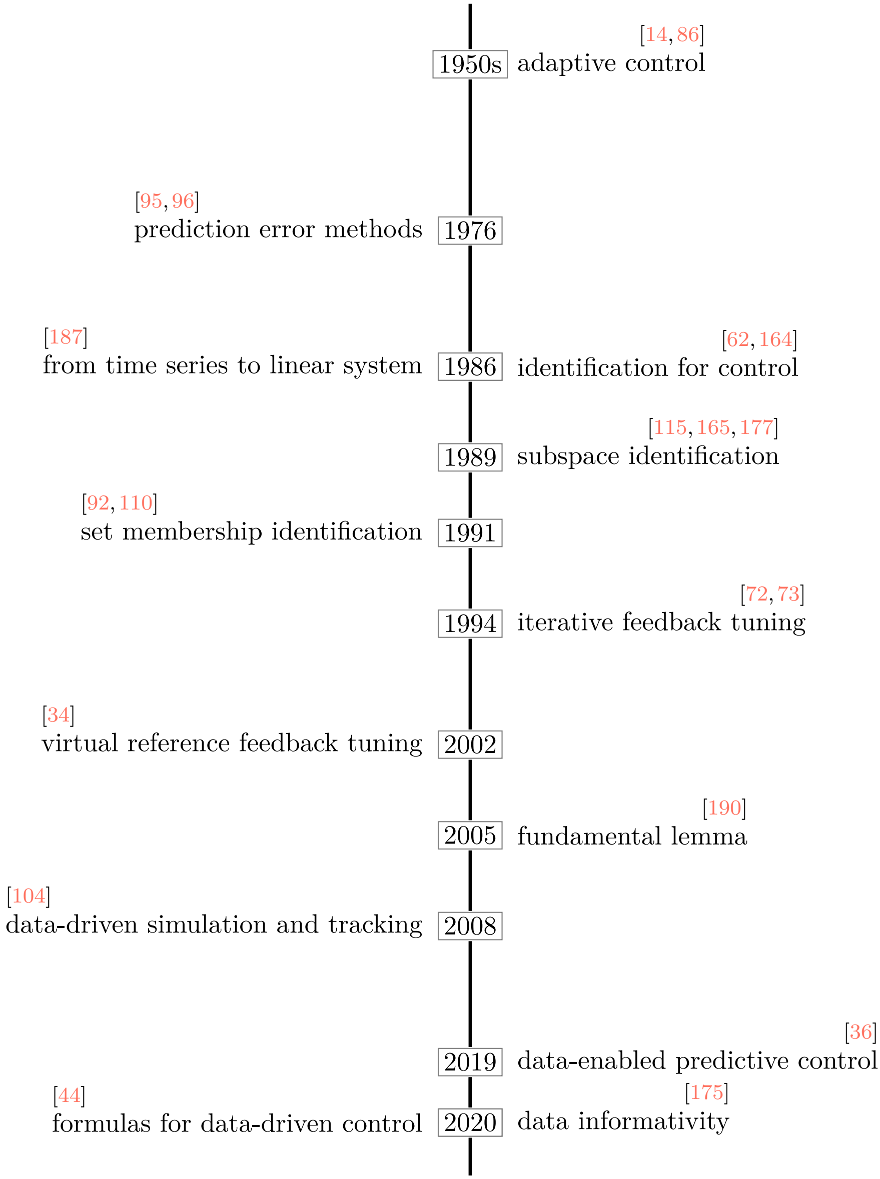


Figure 1.1: Coarse timeline of data-driven modeling and control approaches.

Early developments in direct data-driven control are more scattered than their indirect counterparts. We mention *adaptive control* methods [14], for which both direct and indirect methods exist, and whose origins can already be traced back to the 1950s [86]. In the 1990s and early 2000s, a number of direct data-driven control schemes emerged, including *iterative feedback tuning* (IFT) [72,73] and *virtual reference feedback tuning* (VRFT) [34]. These methods both aim at using data to directly minimize a cost function of the control parameters, with the notable distinction that IFT is an iterative approach while VRFT is one-shot.

In 2005, the paper [190] was published, whose main result would later become known as the *fundamental lemma*. Roughly speaking, the result asserts that all finite-length trajectories of a controllable linear system can be obtained from a single one whose input is persistently exciting. This result has major consequences for the subspace identification methods developed in the 1990s, because it provides conditions on the input data that enable system identification. The fundamental lemma was not widely adopted in the years following its publication, although an early paper is [104] in which the result was used for *direct data-driven simulation and tracking*.

It was only around 2018–2019 that direct data-driven control started to gain a lot of momentum. On the one hand, it is safe to say that the wave of new results was partially due to a renewed interest in the fundamental lemma, that served as a source of inspiration for many new developments. On the other hand, the interest was motivated by the development of low cost sensing devices, meaning that data were by now widely available. This, combined with an increase in available computational power to analyze large datasets, fueled the interest in direct data-driven control.

Because of the importance of the fundamental lemma, we will spend some time to review it in detail in Subsection 1.2.1. We will then highlight the importance of this result in a number of applications ranging from *subspace identification* [115] to *data-driven tracking* [104], *predictive control* [36], and *feedback design* [44]. Finally, we close this chapter with Subsection 1.2.6 by giving a summary of further developments within direct data-driven control in the time period between 2018 and 2025.

1.2.1 The fundamental lemma

In this section we review the fundamental lemma of [190]. To improve the readability of this chapter, we will introduce some notation on the fly throughout. See Chapter A in the appendix for a full account of the notation used in this book. Consider the linear time-invariant (LTI) system

$$x(t+1) = A_{\text{true}}x(t) + B_{\text{true}}u(t) \tag{1.1a}$$
$$y(t) = C_{\text{true}}x(t) + D_{\text{true}}u(t) \tag{1.1b}$$

where $t \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$, $x(t) \in \mathbb{R}^{n_{\text{true}}}$ is the state, $u(t) \in \mathbb{R}^m$ is the input, and $y(t) \in \mathbb{R}^p$ is the output. The true state-space dimension n_{true} and the matrices $A_{\text{true}} \in \mathbb{R}^{n_{\text{true}} \times n_{\text{true}}}$, $B_{\text{true}} \in \mathbb{R}^{n_{\text{true}} \times m}$, $C_{\text{true}} \in \mathbb{R}^{p \times n_{\text{true}}}$ and $D_{\text{true}} \in \mathbb{R}^{p \times m}$ are assumed to be unknown. However, an upper bound N on the state-space dimension is given, i.e., $N \geqslant n_{\text{true}}$. The high-level goal is to simulate and/or control the dynamics of (1.1) using input-output data.

Before we introduce the data, we discuss some preliminaries on *trajectories* of (1.1). A sequence

$$(u(t), x(t), y(t))_{t=0}^{\infty}$$

is called an *input-state-output trajectory* of (1.1) if

$$\begin{bmatrix} x(t+1) \\ y(t) \end{bmatrix} = \begin{bmatrix} A_{\text{true}} & B_{\text{true}} \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

for all $t \in \mathbb{Z}_+$. Since the system (1.1) is *linear*, sums and scalar multiples of input-state-output trajectories are also input-state-output trajectories of (1.1)[1]. Moreover, by *time-invariance* of (1.1), we have that

$$(u(t+\tau), x(t+\tau), y(t+\tau))_{t=0}^{\infty}$$

is also an input-state-output trajectory of (1.1) for any $\tau \in \mathbb{Z}_+$.

A sequence $(u(t), y(t))_{t=0}^{\infty}$ is called an *input-output trajectory* of (1.1) if there exists $x : \mathbb{Z}_+ \to \mathbb{R}^{n_{\text{true}}}$ such that $(u(t), x(t), y(t))_{t=0}^{\infty}$ is an input-state-output trajectory of (1.1). Let $i, j \in \mathbb{Z}_+$ with $i \leqslant j$. Given an input-state-output trajectory $(u(t), x(t), y(t))_{t=0}^{\infty}$, the sequence $(u(t), x(t), y(t))_{t=i}^{j}$ is called a *restricted* input-state-output trajectory (on the time interval $[i, j] := \{i, i+1, \dots, j\}$). Restricted input-output trajectories are defined analogously.

We will identify restricted input-state-output trajectories with the vectors

$$u_{[i,j]} := \begin{bmatrix} u(i) \\ u(i+1) \\ \vdots \\ u(j) \end{bmatrix}, \quad x_{[i,j]} := \begin{bmatrix} x(i) \\ x(i+1) \\ \vdots \\ x(j) \end{bmatrix}, \quad \text{and} \quad y_{[i,j]} := \begin{bmatrix} y(i) \\ y(i+1) \\ \vdots \\ y(j) \end{bmatrix}.$$

[1] In fact, the space of all such trajectories is called the *behavior* of the system [129]. In order to keep the exposition as simple as possible at this point, we do not use behaviors here. However, they will be used at a later stage, in Chapter 9.

We will sometimes also collect restricted trajectories in matrices, which we will denote by capital letters. For example,

$$X_{[i,j]} := \begin{bmatrix} x(i) & x(i+1) & \cdots & x(j) \end{bmatrix}$$

and the matrices $U_{[i,j]}$ and $Y_{[i,j]}$ are defined analogously.

Now, as our *data set*, we consider the restricted input-output trajectory $(u_{[0,T-1]}, y_{[0,T-1]})$ of (1.1), where T is a positive integer. An important ingredient of the fundamental lemma is the Hankel matrix of these inputs and outputs. For a given integer $L \in [1, T]$, let the *Hankel matrix* of depth L of these inputs and outputs be given by:

$$\begin{bmatrix} H_L(u_{[0,T-1]}) \\ \hdashline H_L(y_{[0,T-1]}) \end{bmatrix} = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-L) \\ \vdots & \vdots & & \vdots \\ u(L-1) & u(L) & \cdots & u(T-1) \\ \hdashline y(0) & y(1) & \cdots & y(T-L) \\ \vdots & \vdots & & \vdots \\ y(L-1) & y(L) & \cdots & y(T-1) \end{bmatrix}. \tag{1.2}$$

By time-invariance of the system, each column of (1.2) gives rise to a restricted input-output trajectory of (1.1) on the time interval $[0, L-1]$. Moreover, by linearity of the system, every linear combination of the columns of (1.2) is also a restricted input-output trajectory on the time interval $[0, L-1]$.

The powerful crux of Willems *et al.*'s fundamental lemma is that *every* restricted input-output trajectory of length L can be expressed as a linear combination of the columns of (1.2), assuming that the system (1.1) is controllable, and $u_{[0,T-1]}$ is *persistently exciting* of sufficiently high order. In order to state the result, we first define the concept of persistency of excitation.

Definition 1.1. Let $k \in [1, T]$ be an integer. The input $u_{[0,T-1]}$ is called *persistently exciting of order* k if $H_k(u_{[0,T-1]})$ has full row rank.

Next, we will formulate the fundamental lemma [190].

Theorem 1.2. *Assume that the pair* $(A_{\text{true}}, B_{\text{true}})$ *is controllable. Consider a restricted input-state-output trajectory* $(u_{[0,T-1]}, x_{[0,T-1]}, y_{[0,T-1]})$ *of* (1.1)*, where* $T \geqslant 1$*. Let* $L \in [1, T]$ *be an integer. If the input* $u_{[0,T-1]}$ *is persistently exciting of order* $N + L$*, then the following statements hold:*

(a) *The matrix*

$$\begin{bmatrix} X_{[0,T-L]} \\ \hdashline H_L(u_{[0,T-1]}) \end{bmatrix} = \begin{bmatrix} x(0) & x(1) & \cdots & x(T-L) \\ \hdashline u(0) & u(1) & \cdots & u(T-L) \\ \vdots & \vdots & & \vdots \\ u(L-1) & u(L) & \cdots & u(T-1) \end{bmatrix} \tag{1.3}$$

has full row rank.

(b) $(\bar{u}_{[0,L-1]}, \bar{y}_{[0,L-1]})$ *is a restricted input-output trajectory of* (1.1) *on the time interval* $[0, L-1]$ *if and only if*

$$\begin{bmatrix} \bar{u}_{[0,L-1]} \\ \bar{y}_{[0,L-1]} \end{bmatrix} = \begin{bmatrix} H_L(u_{[0,T-1]}) \\ H_L(y_{[0,T-1]}) \end{bmatrix} g \tag{1.4}$$

for some vector $g \in \mathbb{R}^{T-L+1}$.

(c) *For* $i \in \mathbb{Z}_+$, $(\bar{u}_{[i,i+L-1]}, \bar{y}_{[i,i+L-1]})$ *is a restricted input-output trajectory of* (1.1) *on the time interval* $[i, i+L-1]$ *if and only if*

$$\begin{bmatrix} \bar{u}_{[i,i+L-1]} \\ \bar{y}_{[i,i+L-1]} \end{bmatrix} = \begin{bmatrix} H_L(u_{[0,T-1]}) \\ H_L(y_{[0,T-1]}) \end{bmatrix} g \tag{1.5}$$

for some $g \in \mathbb{R}^{T-L+1}$.

We note that the condition of persistency of excitation requires a sufficiently long trajectory, namely

$$T \geqslant (m+1)(N+L) - 1. \tag{1.6}$$

Statement (a) has appeared first in [190, Cor. 2(iii)]. This result is intriguing since a rank condition on *both* input and state matrices can be imposed by injecting a sufficiently exciting input sequence. Statement (b) is a reformulation of [190, Thm. 1]. We note that the 'if' part of this statement simply follows from the discussion before the theorem. However, the importance of the result lies in the 'only if' part of statement (b), which implies that the image of the Hankel matrix (1.2) is precisely equal to the space of all restricted input-output trajectories on the interval $[0, L-1]$. Finally, note that statement (c) simply boils down to statement (b) in the case that $i = 0$. However, this statement asserts, in addition, that for *any* $i \in \mathbb{Z}_+$, the space of all restricted input-output trajectories on the interval $[i, i+L-1]$ coincides with the image of (1.2). This statement follows from the controllability of $(A_{\text{true}}, B_{\text{true}})$ since, in this case, the spaces of restricted input-output trajectories on the intervals $[i, i+L-1]$ are equal[2] for all $i \in \mathbb{Z}_+$.

Item (a) of the fundamental lemma is the most involved statement to prove. However, as we will see, this statement can be obtained as a corollary of the results in Chapter 11. For this reason, we will postpone the entire proof of Theorem 1.2 to Chapter 11, see Section 11.8. Instead, at this point we note that the fundamental lemma has important consequences for subspace identification and data-driven control applications, which we will review in the next subsections.

1.2.2 Subspace identification

Subspace identification deals with the identification of the state-space dimension n_{true} of the true system (1.1) and the matrices A_{true}, B_{true}, C_{true} and D_{true} from data. In this section, we review a subspace identification result by Moonen *et al.* that was developed in 1989 in [115]. Although the fundamental lemma emerged around fifteen years after this publication, it has important consequences for subspace identification.

The problem of subspace identification is formulated as follows.

Problem 1.3. Consider the system (1.1) and assume that $(A_{\text{true}}, B_{\text{true}})$ is controllable and $(C_{\text{true}}, A_{\text{true}})$ is observable. Given

(a) an upper bound $N \geqslant n_{\text{true}}$ on the state-space dimension of (1.1), and

(b) a restricted input-output trajectory $(u_{[0,T-1]}, y_{[0,T-1]})$ of (1.1),

find the state-space dimension n_{true} of (1.1), and matrices $A \in \mathbb{R}^{n_{\text{true}} \times n_{\text{true}}}$, $B \in \mathbb{R}^{n_{\text{true}} \times m}$, $C \in \mathbb{R}^{p \times n_{\text{true}}}$ and $D \in \mathbb{R}^{p \times m}$ such that

$$A = SA_{\text{true}}S^{-1}, \;\; B = SB_{\text{true}}, \;\; C = C_{\text{true}}S^{-1}, \quad \text{and } D = D_{\text{true}} \tag{1.7}$$

for some nonsingular matrix $S \in \mathbb{R}^{n_{\text{true}} \times n_{\text{true}}}$.

In other words, subspace identification deals with the reconstruction of the true system matrices up to a similarity transformation, using input-output data and an upper bound on the state-space dimension. If (1.8) holds for some nonsingular S then we call the systems (A, B, C, D) and $(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})$ *isomorphic*.

To explain the approach of [115], we start with the following thought experiment. Suppose that, in addition to the input-output data, we also have access to n_{true} and a state sequence $x_{[0,T]}$ that is *consistent with the data*, i.e.,

$$\begin{bmatrix} X_{[1,T]} \\ Y_{[0,T-1]} \end{bmatrix} = \begin{bmatrix} A_{\text{true}} & B_{\text{true}} \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix} \begin{bmatrix} X_{[0,T-1]} \\ U_{[0,T-1]} \end{bmatrix}. \tag{1.8}$$

[2] We note that this is not the case for uncontrollable systems, which may be verified using the example $A_{\text{true}} = 0$, $B_{\text{true}} = 0$, $C_{\text{true}} = 1$ and $D_{\text{true}} = 0$.

If the matrix

$$\begin{bmatrix} X_{[0,T-1]} \\ U_{[0,T-1]} \end{bmatrix}$$

has full row rank, then the linear equation (1.8) has a unique solution

$$(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}}).$$

In this situation, we can thus uniquely identify the true system from input-state-output data.

Of course, the state sequence $x_{[0,T]}$ and its dimension are not part of the data, so the above approach cannot be applied directly. However, a central idea[3] in the subspace identification literature is to *identify a state sequence* from input-output data. Since the data $(u_{[0,T-1]}, y_{[0,T-1]})$ can be produced by any system that is isomorphic to $(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})$, the best we can hope for is to identify n_{true} and $SX_{[0,T]}$ for some nonsingular matrix $S \in \mathbb{R}^{n_{\text{true}} \times n_{\text{true}}}$, i.e., to identify the *row space*[4] of the matrix of states $X_{[0,T]}$.

To this end, the paper [115] assumes that $T \geqslant 2N$ and considers the following partitioned input Hankel matrix:

$$H_{2N}(u_{[0,T-1]}) = \left[\begin{array}{cccc} u(0) & u(1) & \cdots & u(T-2N) \\ \vdots & \vdots & & \vdots \\ u(N-1) & u(N) & \cdots & u(T-N-1) \\ \hdashline u(N) & u(N+1) & \cdots & u(T-N) \\ \vdots & \vdots & & \vdots \\ u(2N-1) & u(2N) & \cdots & u(T-1) \end{array}\right] =: \left[\begin{array}{c} U_p \\ \hdashline U_f \end{array}\right].$$

The Hankel matrix $H_{2N}(y_{[0,T-1]})$ of outputs is partitioned similarly into the blocks Y_p and Y_f. The matrices U_p and Y_p are often referred to as 'past' data matrices, while U_f and Y_f are 'future' data matrices. This terminology should be taken with a grain of salt since some inputs (like $u(N)$) appear in both U_p and U_f. With this terminology in place, we state the main result of [115].

Proposition 1.4 (Theorem 3 of [115])**.** *Assume that* $(A_{\text{true}}, B_{\text{true}})$ *is controllable and* $(C_{\text{true}}, A_{\text{true}})$ *is observable. Let* $(u_{[0,T-1]}, x_{[0,T-1]}, y_{[0,T-1]})$ *be a restricted input-state-output trajectory of system* (1.1)*. Assume that the following*

[3] There are also other subspace identification methods that aim at first reconstructing the observability matrix from data, see [165] for an overview.

[4] The *row space* of a matrix M is the space of all linear combinations of the rows of M, and is denoted by $\operatorname{rsp} M$.

rank conditions hold:

$$\operatorname{rank}\begin{bmatrix} H_{2N}(u_{[0,T-1]}) \\ H_{2N}(y_{[0,T-1]}) \end{bmatrix} = n_{\text{true}} + 2Nm, \ \operatorname{rank}\begin{bmatrix} U_p \\ Y_p \end{bmatrix} = \operatorname{rank}\begin{bmatrix} U_f \\ Y_f \end{bmatrix} = n_{\text{true}} + Nm. \tag{1.9}$$

Then $X_{[N,T-N]}$ *has rank* n_{true} *and its row space is given by:*

$$\operatorname{rsp} X_{[N,T-N]} = \operatorname{rsp}\begin{bmatrix} U_p \\ Y_p \end{bmatrix} \cap \operatorname{rsp}\begin{bmatrix} U_f \\ Y_f \end{bmatrix}. \tag{1.10}$$

In other words, under the three rank conditions in (1.9), we can obtain the row space of the state sequence on the time interval $[N, T-N]$ from input-output data. At the time of [115], it was not very well-understood how the rank conditions (1.9) can be verified and/or imposed. In fact, at first glance, it appears to be impossible to verify (1.9) since n_{true} is unknown. Indeed, we recall that only an upper bound N on n_{true} is given. An important consequence of the fundamental lemma is that (1.9) can be *imposed* by choosing the input sequence to be sufficiently persistently exciting, after which n_{true} can be extracted from the data. In fact, the following proposition follows in a straightforward manner from the fundamental lemma.

Proposition 1.5. *Assume that* $(A_{\text{true}}, B_{\text{true}})$ *is controllable and* $(C_{\text{true}}, A_{\text{true}})$ *is observable. Let*

$$(u_{[0,T-1]}, x_{[0,T-1]}, y_{[0,T-1]})$$

be a restricted input-state-output trajectory of system (1.1). If $u_{[0,T-1]}$ *is persistently exciting of order* $3N$ *then (1.9) holds.*

For example, the first rank condition of (1.9) follows from the fact that

$$\begin{bmatrix} H_{2N}(u_{[0,T-1]}) \\ H_{2N}(y_{[0,T-1]}) \end{bmatrix} = \begin{bmatrix} 0 & I \\ \Omega_{2N} & \Theta_{2N} \end{bmatrix} \begin{bmatrix} X_{[0,T-2N]} \\ H_{2N}(u_{[0,T-1]}) \end{bmatrix}, \tag{1.11}$$

where Ω_{2N} and Θ_{2N} are observability and Toeplitz matrices, defined recursively via

$$\Omega_1 = C_{\text{true}}, \ \Omega_{k+1} = \begin{bmatrix} \Omega_k \\ C_{\text{true}} A_{\text{true}}^k \end{bmatrix} \tag{1.12}$$

$$\Gamma_1 = B_{\text{true}}, \ \Gamma_{k+1} = \begin{bmatrix} A_{\text{true}}^k B_{\text{true}} & \Gamma_k \end{bmatrix} \tag{1.13}$$

$$\Theta_1 = D_{\text{true}}, \ \Theta_{k+1} = \begin{bmatrix} \Theta_k & 0 \\ C_{\text{true}}\Gamma_k & D_{\text{true}} \end{bmatrix}, \tag{1.14}$$

for $k \geqslant 1$. Since the pair $(C_{\text{true}}, A_{\text{true}})$ is observable and $2N \geqslant n_{\text{true}}$, the matrix Ω_{2N} has rank n_{true}. As such, the matrix

$$\begin{bmatrix} 0 & I \\ \Omega_{2N} & \Theta_{2N} \end{bmatrix}$$

has full column rank. We conclude from (1.11) that

$$\operatorname{rank}\begin{bmatrix} H_{2N}(u_{[0,T-1]}) \\ H_{2N}(y_{[0,T-1]}) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} X_{[0,T-2N]} \\ H_{2N}(u_{[0,T-1]}) \end{bmatrix} = n_{\text{true}} + 2Nm,$$

where the last equality follows from Theorem 1.2 and the assumption that $u_{[0,T-1]}$ is persistently exciting of order $3N$, and thus also of order $n_{\text{true}} + 2N$. The other two rank conditions in (1.9) can be proven in a similar way.

Now, note that under the condition of persistency of excitation of order $3N$, Propositions 1.4 and 1.5 allow us to extract n_{true} from input-output data as

$$n_{\text{true}} = \dim\left(\operatorname{rsp}\begin{bmatrix} U_p \\ Y_p \end{bmatrix} \cap \operatorname{rsp}\begin{bmatrix} U_f \\ Y_f \end{bmatrix}\right).$$

Moreover, we can obtain vectors $\bar{x}(N), \bar{x}(N+1), \ldots, \bar{x}(T-N) \in \mathbb{R}^{n_{\text{true}}}$ from the input-output data such that the matrix

$$\bar{X}_{[N,T-N]} := \begin{bmatrix} \bar{x}(N) & \bar{x}(N+1) & \cdots & \bar{x}(T-N) \end{bmatrix}$$

satisfies

$$\operatorname{rsp} \bar{X}_{[N,T-N]} = \operatorname{rsp}\begin{bmatrix} U_p \\ Y_p \end{bmatrix} \cap \operatorname{rsp}\begin{bmatrix} U_f \\ Y_f \end{bmatrix}.$$

Then, by Proposition 1.4, $SX_{[N,T-N]} = \bar{X}_{[N,T-N]}$ for some nonsingular matrix $S \in \mathbb{R}^{n_{\text{true}} \times n_{\text{true}}}$. We then conclude that

$$\begin{bmatrix} \bar{X}_{[N+1,T-N]} \\ Y_{[N,T-N-1]} \end{bmatrix} = \begin{bmatrix} SA_{\text{true}}S^{-1} & SB_{\text{true}} \\ C_{\text{true}}S^{-1} & D_{\text{true}} \end{bmatrix} \begin{bmatrix} \bar{X}_{[N,T-N-1]} \\ U_{[N,T-N-1]} \end{bmatrix}.$$

Therefore, if the matrix

$$\begin{bmatrix} \bar{X}_{[N,T-N-1]} \\ U_{[N,T-N-1]} \end{bmatrix}$$

has full row rank[5], then the system of linear equations

$$\begin{bmatrix} \bar{X}_{[N+1,T-N]} \\ Y_{[N,T-N-1]} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \bar{X}_{[N,T-N-1]} \\ U_{[N,T-N-1]} \end{bmatrix}$$

has a unique solution (A, B, C, D) and, moreover, $A = SA_{\text{true}}S^{-1}$, $B = SB_{\text{true}}$, $C = C_{\text{true}}S^{-1}$, and $D = D_{\text{true}}$. This provides a solution to the subspace identification problem.

There is a lot more that can be said about the topic of subspace identification. Our aim in this subsection was only to explain some of the basic ideas

[5] Also this rank condition can be imposed, in conjunction with (1.9), by choosing $u_{[0,T-2]}$ to be persistently exciting of order $3N$.

and to make a connection to the fundamental lemma. For further extensions, for example to the case of noisy data, we refer to the books [165,178]. In the context of the current book, it is interesting to note that the condition of persistency of excitation of order $3N$ is sufficient but *not necessary* to solve the subspace identification problem. In fact, it is quite straightforward to see that persistency of excitation of order $2N+1$ is sufficient, by noting that all input-output trajectories of (1.1) can be obtained from all restricted input-output trajectories defined on the interval $[0, N]$. However, even this condition is not necessary in general. Necessary and sufficient conditions under which the input-output data contain enough information to identify the system matrices were described in the paper [32]. We will come back to this point in Chapter 11, where we treat such necessary and sufficient conditions in detail.

1.2.3 Data-driven simulation and tracking

A few years after the publication of the fundamental lemma, in [104] a framework was proposed for data-driven simulation and control based on the parameterization of trajectories as expressed by (1.5). In this subsection, we will review some of the ideas from their paper. We will start with the problem of simulating trajectories of (1.1) using measured input-output data. This problem is formalized as follows.

Problem 1.6. Consider the system (1.1). Assume that $(A_{\text{true}}, B_{\text{true}})$ is controllable. Let L_{ini} and L_{ref} be positive integers and define $L := L_{\text{ini}} + L_{\text{ref}}$. Given

(a) an upper bound $N \geqslant n_{\text{true}}$ on the state-space dimension of (1.1),

(b) a restricted input-output trajectory $(u_{[0,T-1]}, y_{[0,T-1]})$ of (1.1) with $T \geqslant L$,

(c) an *initial restricted trajectory* $(\bar{u}_{[0,L_{\text{ini}}-1]}, \bar{y}_{[0,L_{\text{ini}}-1]})$ of (1.1), and

(d) a *reference input* $\bar{u}_{[L_{\text{ini}},L-1]} \in \mathbb{R}^{mL_{\text{ref}}}$,

find outputs $\bar{y}_{[L_{\text{ini}},L-1]}$ such that $(\bar{u}_{[0,L-1]}, \bar{y}_{[0,L-1]})$ is a restricted input-output trajectory of (1.1).

So the problem is to use the data $(u_{[0,T-1]}, y_{[0,T-1]})$ to find ('simulate') the output $\bar{y}_{[L_{\text{ini}},L-1]}$, given the reference input $\bar{u}_{[L_{\text{ini}},L-1]}$ and an initial trajectory of the system. The reason for including the initial trajectory is to 'fix' an initial state of the system (1.1). Indeed, given only the reference input $\bar{u}_{[L_{\text{ini}},L-1]}$, the output $\bar{y}_{[L_{\text{ini}},L-1]}$ is not unique, since it also depends on the state of system (1.1) at time L_{ini}. We will see, however, that by choosing the length of the initial trajectory appropriately, there exists a *unique* output sequence $\bar{y}_{[L_{\text{ini}},L-1]}$ which can be readily computed from the data.

We will now outline the approach of [104] to solve Problem 1.6. To this end, we partition the Hankel matrices of the input-output data in a similar way as in Section 1.2.2. In particular,

$$H_L(u_{[0,T-1]}) = \begin{bmatrix} U_p \\ U_f \end{bmatrix} \tag{1.15}$$

$$H_L(y_{[0,T-1]}) = \begin{bmatrix} Y_p \\ Y_f \end{bmatrix} \tag{1.16}$$

where U_p and Y_p have mL_{ini} and pL_{ini} rows, and U_f and Y_f have mL_{ref} and pL_{ref} rows, respectively. Moreover, we define the *lag* ℓ_{true} of (1.1) as the smallest integer k for which $\operatorname{rank}\Omega_k = \operatorname{rank}\Omega_{k+1}$, where we recall that Ω_k is the observability matrix defined in (1.12). With this in mind, we state the following theorem from [104].

Theorem 1.7. *Consider the system (1.1) and assume that $(A_{\text{true}}, B_{\text{true}})$ is controllable. Suppose that $u_{[0,T-1]}$ is persistently exciting of order $N+L$. Moreover, consider the restricted input-output trajectory $(\bar{u}_{[0,L_{\text{ini}}-1]}, \bar{y}_{[0,L_{\text{ini}}-1]})$ of (1.1) and the reference input $\bar{u}_{[L_{\text{ini}},L-1]} \in \mathbb{R}^{mL_{\text{ref}}}$. Then the following statements hold:*

(a) *The system of equations*

$$\begin{bmatrix} U_p \\ Y_p \\ U_f \end{bmatrix} g = \begin{bmatrix} \bar{u}_{[0,L_{\text{ini}}-1]} \\ \bar{y}_{[0,L_{\text{ini}}-1]} \\ \bar{u}_{[L_{\text{ini}},L-1]} \end{bmatrix} \tag{1.17}$$

has at least one solution $g \in \mathbb{R}^{T-L+1}$.

(b) *Let g be a solution to (1.17) and define the output $\bar{y}_{[L_{\text{ini}},L-1]} := Y_f g$. Then $(\bar{u}_{[0,L-1]}, \bar{y}_{[0,L-1]})$ is a restricted input-output trajectory of (1.1).*

(c) *If, in addition, $L_{\text{ini}} \geqslant \ell_{\text{true}}$ then the output $\bar{y}_{[L_{\text{ini}},L-1]}$ in (b) is unique in the sense that it is the only vector in $\mathbb{R}^{pL_{\text{ref}}}$ for which $(\bar{u}_{[0,L-1]}, \bar{y}_{[0,L-1]})$ is a restricted input-output trajectory of (1.1).*

Theorem 1.7 provides a simple approach to simulate the output of the dynamical system (1.1) by solving a system of equations (1.17), where the coefficient matrix is constructed directly from a Hankel matrix of input-output data. Depending on the application, the initial trajectory $(\bar{u}_{[0,L_{\text{ini}}-1]}, \bar{y}_{[0,L_{\text{ini}}-1]})$ can be chosen in different ways. For example, in [104, Sec. 4.5], Theorem 1.7 is used to simulate the first L_{ref} *Markov parameters* of the system (1.1), i.e., the matrices

D_{true} and $C_{\text{true}}A^i_{\text{true}}B_{\text{true}}$ for $i = 0, 1, \dots, L_{\text{ref}} - 2$. This is done by simulating m input-output trajectories, where in each trajectory, $\bar{u}_{[0,L_{\text{ini}}-1]} = 0$ and $\bar{y}_{[0,L_{\text{ini}}-1]} = 0$. The reference input is selected as an 'impulse', i.e., $\bar{u}(L_{\text{ini}}) = e_i$ and $\bar{u}(L_{\text{ini}}+1) = \cdots = \bar{u}(L-1) = 0$, where e_i is the i-th standard basis vector of $\mathbb{R}^m$ for $i = 1, 2, \dots, m$. This ensures that the i-th simulated output trajectory is equal to the i-th column of a matrix containing the first L_{ref} Markov parameters of (1.1), see [104, Prop. 11].

In addition to the simulation problem, [104] also considers data-driven tracking. In what follows, we will review this problem in more detail.

Problem 1.8. Consider the system (1.1). Assume that $(A_{\text{true}}, B_{\text{true}})$ is controllable. Let L_{ini} and L_{ref} be positive integers and define $L := L_{\text{ini}} + L_{\text{ref}}$. Given

(a) an upper bound $N \geqslant n_{\text{true}}$ on the state-space dimension of (1.1),

(b) a symmetric positive semidefinite matrix[6]

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

where $Q_{11} \in \mathbb{R}^{m\times m}$, $Q_{12} = Q_{21}^\top \in \mathbb{R}^{m\times p}$, and $Q_{22} \in \mathbb{R}^{p\times p}$.

(c) a restricted input-output trajectory $(u_{[0,T-1]}, y_{[0,T-1]})$ of (1.1) with $T \geqslant L$,

(d) an *initial restricted trajectory* $(\bar{u}_{[0,L_{\text{ini}}-1]}, \bar{y}_{[0,L_{\text{ini}}-1]})$ of (1.1), and

(e) a *reference signal* $(v_{[L_{\text{ini}},L-1]}, z_{[L_{\text{ini}},L-1]}) \in \mathbb{R}^{mL_{\text{ref}}} \times \mathbb{R}^{pL_{\text{ref}}}$,

find $(\bar{u}_{[L_{\text{ini}},L-1]}, \bar{y}_{[L_{\text{ini}},L-1]})$ that minimizes the cost function

$$\sum_{t=L_{\text{ini}}}^{L-1} \begin{bmatrix} \bar{u}(t) - v(t) \\ \bar{y}(t) - z(t) \end{bmatrix}^\top Q \begin{bmatrix} \bar{u}(t) - v(t) \\ \bar{y}(t) - z(t) \end{bmatrix} \tag{1.18}$$

subject to the constraint that $(\bar{u}_{[0,L-1]}, \bar{y}_{[0,L-1]})$ is a restricted input-output trajectory of (1.1).

The paper [104] focuses on the case that Q is positive definite. In this case, three solutions are proposed for Problem 1.8. Two of them are indirect methods that first compute a representation of the system (1.1) in the form of a state-space model or impulse response matrix. The other one is a direct approach

[6] A symmetric matrix $M \in \mathbb{R}^{n\times n}$ is called *positive semidefinite* if $x^\top Mx \geqslant 0$ for all $x \in \mathbb{R}^n$ and *positive definite* if $x^\top Mx > 0$ for all nonzero $x \in \mathbb{R}^n$. This is denoted by $M \geqslant 0$ and $M > 0$, respectively.

that uses concepts from the behavioral approach to systems and control. In this subsection, we formulate an alternative solution to Problem 1.8 that relies on elementary concepts like the solution to a constrained least squares problem.

As in the simulation problem, we will consider the partitioned Hankel matrices of the data in (1.15) and (1.16). Now, we note that if $u_{[0,T-1]}$ is persistently exciting of order $N+L$, then, by Theorem 1.2, the constraint of the minimization problem in Problem 1.8 is equivalent to the existence of $g \in \mathbb{R}^{T-L+1}$ such that

$$\begin{bmatrix} \bar{u}_{[0,L_{\text{ini}}-1]} \\ \bar{y}_{[0,L_{\text{ini}}-1]} \\ \bar{u}_{[L_{\text{ini}},L-1]} \\ \bar{y}_{[L_{\text{ini}},L-1]} \end{bmatrix} = \begin{bmatrix} U_p \\ Y_p \\ U_f \\ Y_f \end{bmatrix} g. \tag{1.19}$$

We define the matrix

$$\bar{Q} := \begin{bmatrix} I_{L_{\text{ref}}} \otimes Q_{11} & I_{L_{\text{ref}}} \otimes Q_{12} \\ I_{L_{\text{ref}}} \otimes Q_{21} & I_{L_{\text{ref}}} \otimes Q_{22} \end{bmatrix} \tag{1.20}$$

where $\otimes$ denotes the Kronecker product. With this notation in place, the cost function in (1.18) can be rewritten as

$$\left\| \bar{Q}^{\frac{1}{2}} \begin{bmatrix} \bar{u}_{[L_{\text{ini}},L-1]} \\ \bar{y}_{[L_{\text{ini}},L-1]} \end{bmatrix} - \bar{Q}^{\frac{1}{2}} \begin{bmatrix} v_{[L_{\text{ini}},L-1]} \\ z_{[L_{\text{ini}},L-1]} \end{bmatrix} \right\|^2 .$$

Then, Problem 1.8 can be reformulated as the problem of finding a minimizer $g \in \mathbb{R}^{T-L+1}$ of the optimization problem

$$\begin{aligned} &\text{minimize} && \left\| \bar{Q}^{\frac{1}{2}} \begin{bmatrix} U_f \\ Y_f \end{bmatrix} g - \bar{Q}^{\frac{1}{2}} \begin{bmatrix} v_{[L_{\text{ini}},L-1]} \\ z_{[L_{\text{ini}},L-1]} \end{bmatrix} \right\|^2 \\ &\text{subject to} && \begin{bmatrix} U_p \\ Y_p \end{bmatrix} g = \begin{bmatrix} \bar{u}_{[0,L_{\text{ini}}-1]} \\ \bar{y}_{[0,L_{\text{ini}}-1]} \end{bmatrix}. \end{aligned} \tag{1.21}$$

Indeed, for any such minimizer g, the trajectory $(\bar{u}_{[L_{\text{ini}},L-1]}, \bar{y}_{[L_{\text{ini}},L-1]})$ defined by

$$\begin{bmatrix} \bar{u}_{[L_{\text{ini}},L-1]} \\ \bar{y}_{[L_{\text{ini}},L-1]} \end{bmatrix} := \begin{bmatrix} U_f \\ Y_f \end{bmatrix} g$$

is a solution to Problem 1.8. Vice versa, given a solution $(\bar{u}_{[L_{\text{ini}},L-1]}, \bar{y}_{[L_{\text{ini}},L-1]})$ to Problem 1.8, any solution g to (1.19) is a minimizer of (1.21).

Next, we note that (1.21) is a least squares problem with a linear equality constraint. The following basic lemma discusses conditions under which such a problem has a solution, and how to find one if it exists. For additional information on constrained least squares problems, we refer to [27, Ch. 16].

Lemma 1.9. *Let $A \in \mathbb{R}^{n\times m}$, $b \in \mathbb{R}^n$, $C \in \mathbb{R}^{k\times m}$ and $d \in \mathbb{R}^k$. Consider the problem*

$$\begin{aligned} &\text{minimize } \|Ax-b\|^2 \\ &\text{subject to } Cx = d \text{ and } x \in \mathbb{R}^m. \end{aligned} \tag{1.22}$$

Assume that $d \in \operatorname{im} C$. Then, the system of equations

$$\begin{bmatrix} A^\top A & C^\top \\ C & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} A^\top b \\ d \end{bmatrix} \tag{1.23}$$

has at least one solution

$$\begin{bmatrix} \hat{x} \\ \hat{z} \end{bmatrix} \in \mathbb{R}^{m+k}.$$

Moreover, $\hat{x}$ is a minimizer of (1.22) if and only if there exists $\hat{z} \in \mathbb{R}^k$ such that (1.23) holds.

Based on Lemma 1.9, we now formulate the following solution to Problem 1.8.

Theorem 1.10. *Consider the system (1.1) and assume that $(A_{\text{true}}, B_{\text{true}})$ is controllable. Suppose that $u_{[0,T-1]}$ is persistently exciting of order $N+L$. There exist vectors $g \in \mathbb{R}^{T-L+1}$ and $h \in \mathbb{R}^{(m+p)L_{\text{ini}}}$ such that*

$$\left[\begin{array}{c:c} \begin{bmatrix} U_f \\ Y_f \end{bmatrix}^\top \bar{Q} \begin{bmatrix} U_f \\ Y_f \end{bmatrix} & \begin{bmatrix} U_p \\ Y_p \end{bmatrix}^\top \\ \hdashline \begin{bmatrix} U_p \\ U_f \end{bmatrix} & 0 \end{array}\right] \left[\begin{array}{c} g \\ \hdashline h \end{array}\right] = \left[\begin{array}{c} \begin{bmatrix} U_f \\ Y_f \end{bmatrix}^\top \bar{Q} \begin{bmatrix} v_{[L_{\text{ini}},L-1]} \\ z_{[L_{\text{ini}},L-1]} \end{bmatrix} \\ \hdashline \begin{bmatrix} \bar{u}_{[0,L_{\text{ini}}-1]} \\ \bar{y}_{[0,L_{\text{ini}}-1]} \end{bmatrix} \end{array}\right].$$

For such g and h, define

$$\begin{bmatrix} \bar{u}_{[L_{\text{ini}},L-1]} \\ \bar{y}_{[L_{\text{ini}},L-1]} \end{bmatrix} := \begin{bmatrix} U_f \\ Y_f \end{bmatrix} g.$$

Then, $(\bar{u}_{[L_{\text{ini}},L-1]}, \bar{y}_{[L_{\text{ini}},L-1]})$ is a solution to Problem 1.8.

1.2.4 Data-enabled predictive control

In this section we will review the paper [36], in which the fundamental lemma was applied to develop data-enabled predictive controllers for linear systems[7]. The main idea of [36] is to apply data-driven tracking in a receding horizon manner. This means that, at every time step, a sequence of 'predicted' inputs

[7]We note that the idea of using the fundamental lemma for data-driven predictive control design has also been used in the paper [196].

and outputs is computed that solves a finite-horizon tracking problem as studied in Section 1.2.3. Only the first[8] input is applied to the system, after which the procedure is repeated.

For data-enabled predictive control we require the following ingredients:

(a) an upper bound $N \geqslant n_{\text{true}}$ on the state-space dimension of (1.1),

(b) positive integers L_{ini} and L_{ref} with $L_{\text{ini}} \geqslant \ell_{\text{true}}$ and $L := L_{\text{ini}} + L_{\text{ref}}$,

(c) a symmetric positive semidefinite matrix

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

where $Q_{11} \in \mathbb{R}^{m\times m}$, $Q_{12} = Q_{21}^\top \in \mathbb{R}^{m\times p}$ and $Q_{22} \in \mathbb{R}^{p\times p}$,

(d) a restricted input-output trajectory $(u_{[0,T-1]}, y_{[0,T-1]})$ of (1.1), where the input $u_{[0,T-1]}$ persistently exciting of order $N + L$,

(e) an *initial restricted trajectory* $(\bar{u}_{[0,L_{\text{ini}}-1]}, \bar{y}_{[0,L_{\text{ini}}-1]})$ of (1.1),

(f) a *reference signal* $(v(t), z(t))_{t=L_{\text{ini}}}^{\infty}$ where $v(t) \in \mathbb{R}^m$ and $z(t) \in \mathbb{R}^p$ for $t \in [L_{\text{ini}}, \infty)$.

In [36], the authors further work under the assumptions that $Q_{12} = Q_{21}^\top = 0$ and $v(t) = 0$ for all $t \in \mathbb{Z}_+$. With these ingredients in place, we recall the data-enabled predictive control algorithm [36, Alg. 2]. Starting from the initial restricted trajectory $(\bar{u}_{[0,L_{\text{ini}}-1]}, \bar{y}_{[0,L_{\text{ini}}-1]})$ of (1.1), the algorithm computes the inputs $\bar{u}(\tau + L_{\text{ini}})$ in an iterative manner by solving a finite horizon data-driven tracking problem for every $\tau = 0, 1, \ldots$.

procedure DATA-ENABLED PREDICTIVE CONTROL
  require: Positive integers L_{ini} and L_{ref}, positive semidefinite matrix Q, data $(u_{[0,T-1]}, y_{[0,T-1]})$, initial restricted trajectory $(\bar{u}_{[0,L_{\text{ini}}-1]}, \bar{y}_{[0,L_{\text{ini}}-1]})$, and reference signal $(v(t), z(t))_{t=L_{\text{ini}}}^{\infty}$.
  for $\tau = 0, 1, \ldots$ **do**
    Compute

$$u^{\text{pred}}(\tau + L_{\text{ini}}), \ldots, u^{\text{pred}}(\tau + L - 1) \in \mathbb{R}^m$$

    and

$$y^{\text{pred}}(\tau + L_{\text{ini}}), \ldots, y^{\text{pred}}(\tau + L - 1) \in \mathbb{R}^p$$

[8] Alternatively, a number of predicted inputs can be applied to the system before repeating the procedure [36].

that minimize the cost function

$$\sum_{t=\tau+L_{\text{ini}}}^{\tau+L-1} \begin{bmatrix} u^{\text{pred}}(t) - v(t) \\ y^{\text{pred}}(t) - z(t) \end{bmatrix}^\top Q \begin{bmatrix} u^{\text{pred}}(t) - v(t) \\ y^{\text{pred}}(t) - z(t) \end{bmatrix} \tag{1.24}$$

subject to the constraint that

$$\left(\begin{bmatrix} \bar{u}_{[\tau,\tau+L_{\text{ini}}-1]} \\ u^{\text{pred}}_{[\tau+L_{\text{ini}},\tau+L-1]} \end{bmatrix}, \begin{bmatrix} \bar{y}_{[\tau,\tau+L_{\text{ini}}-1]} \\ y^{\text{pred}}_{[\tau+L_{\text{ini}},\tau+L-1]} \end{bmatrix} \right)$$

is a restricted input-output trajectory of (1.1).

Apply $\bar{u}(\tau + L_{\text{ini}}) := u^{\text{pred}}(\tau + L_{\text{ini}})$ to (1.1) and measure $\bar{y}(\tau + L_{\text{ini}})$.

end for

end procedure

The minimization of (1.24) is essentially a finite horizon tracking problem that can be solved in the same way as in Section 1.2.3. In [36], it was shown that data-enabled predictive control is equivalent to model predictive control under the assumptions of the fundamental lemma. More precisely, if $(A_{\text{true}}, B_{\text{true}})$ is controllable and $u_{[0,T-1]}$ is persistently exciting, the above procedure generates the same input-output trajectory as an associated (model-based) model predictive control scheme, given the same initial and reference trajectory. In the case that the data $(u_{[0,T-1]}, y_{[0,T-1]})$ are corrupted by noise, the paper [37] further proposes robust versions of the basic data-enabled predictive control algorithm, by adding regularization terms to the objective function in (1.24). We also note that the paper [19] further studies the stability of data-driven predictive control schemes with terminal constraints.

1.2.5 Formulas for data-driven control

The paper [44] also approaches data-driven control from the perspective of the fundamental lemma. In contrast to the results in the previous sections, however, [44] focuses on the design of state feedback controllers of the form $u(t) = Kx(t)$. In this subsection, we will review the main idea of [44]. For this, we will focus on the input-state dynamics (1.1a) and the input-state data $u_{[0,T-1]}$ and $x_{[0,T]}$, collected from (1.1a). In this case, the state-space dimension n_{true} of (1.1) is assumed to be known. Define the matrices

$$X_- := X_{[0,T-1]}, \;\; X_+ := X_{[1,T]}, \;\; U_- := U_{[0,T-1]}.$$

Then we have the following relation between the true system matrices and the data:

$$X_+ = A_{\text{true}} X_- + B_{\text{true}} U_-. \tag{1.25}$$

If $(A_{\text{true}}, B_{\text{true}})$ is controllable and $u_{[0,T-1]}$ is persistently exciting of order $n_{\text{true}}+1$, then we see that

$$\operatorname{rank}\begin{bmatrix} X_- \\ U_- \end{bmatrix} = n_{\text{true}} + m \tag{1.26}$$

by applying Theorem 1.2 with $L = 1$. Moreover, if (1.26) holds, then for any matrix $K \in \mathbb{R}^{m\times n_{\text{true}}}$, there exists a $G \in \mathbb{R}^{T\times n_{\text{true}}}$ such that

$$\begin{bmatrix} I \\ K \end{bmatrix} = \begin{bmatrix} X_- \\ U_- \end{bmatrix} G. \tag{1.27}$$

This means that the closed-loop system, obtained from interconnecting (1.1a) with the controller $u(t) = Kx(t)$ can be expressed as

$$\begin{aligned} x(t+1) = (A_{\text{true}} + B_{\text{true}}K)x(t) &= \begin{bmatrix} A_{\text{true}} & B_{\text{true}} \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} x(t) \\ &= \begin{bmatrix} A_{\text{true}} & B_{\text{true}} \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} Gx(t) = X_+Gx(t). \end{aligned}$$

The idea of [44] is now to impose suitable properties on the closed-loop system matrix X_+G by choosing G appropriately. As long as such a G satisfies $X_-G = I$, we can retrieve a suitable feedback K as $K = U_-G$, viz. (1.27). In what follows, we will focus on the specific problem of data-driven stabilization, with the goal of finding a G such that the closed-loop system

$$x(t+1) = X_+Gx(t)$$

is stable, equivalently, the matrix X_+G is a stable matrix, meaning that all its eigenvalues have modulus strictly less than one. With this terminology in place, we now recall the following theorem from [44, Thm. 3].

Theorem 1.11. *Suppose that (1.26) holds. If the matrix $\Theta \in \mathbb{R}^{T\times n_{\text{true}}}$ satisfies*

$$X_-\Theta = (X_-\Theta)^\top \text{ and } \begin{bmatrix} X_-\Theta & X_+\Theta \\ \Theta^\top X_+^\top & X_-\Theta \end{bmatrix} > 0 \tag{1.28}$$

then the controller

$$K = U_-\Theta(X_-\Theta)^{-1} \tag{1.29}$$

is such that $A_{\text{true}}+B_{\text{true}}K$ is stable. Conversely, if K is such that $A_{\text{true}}+B_{\text{true}}K$ is stable, then it is of the form (1.29) with Θ a solution to (1.28).

We note that the paper [44] further studies the problem of designing linear quadratic regulators using data, and also discusses the case in which measurements are corrupted by noise. The approach was further extended to other classes of systems in follow-up works, such as the paper [68] on polynomial systems. Theorem 1.11 is relevant in the context of Section 6.1 of this book. In fact, as we will show there, the rank condition (1.26) (and persistency of excitation of the input $u_{[0,T-1]}$) is in general not necessary to find stabilizing controllers using data. Using the data informativity framework developed in this book, we will derive necessary and sufficient conditions on the data under which data-driven stabilization is possible.

1.2.6 Further developments to data-driven control

In addition to the work treated in the previous subsections, a large number of contributions to data-driven control have emerged. Central to this book is the informativity approach to data-driven control [173, 175], which will be introduced in detail in Chapter 2. In contrast to the work discussed above, within the informativity approach we do not assume that the data are persistently exciting. Instead, we aim at deriving necessary and sufficient conditions under which models, system properties, and controllers can be obtained from the data. Such necessary and sufficient conditions provide valuable insight into the relative merits of direct and indirect methods. For example, in Section 6.1 we will see that stabilizing controllers may be found from data that do not enable system identification, thereby highlighting the power of direct data-driven control techniques. The conditions obtained through the informativity framework are also important in situations where generating persistently exciting inputs is challenging, for example, because the number of data points are limited or the data are collected in closed-loop. In the latter scenario, the loss of persistency of excitation is a well-known issue in system identification [163], typically requiring the addition of external reference inputs. As we will see, the informativity approach also naturally applies to data that are corrupted by noise. In this situation, we will provide a plethora of analysis and control design tools allowing us to ascertain system properties and design robust controllers for all systems that are consistent with the data.

The topic of noise-corrupted data has received a lot of attention, and various ways of modeling the noise have been considered. For example, the paper [40] considers process noise of bounded ℓ_∞-norm. This leads to a polytopic set of systems consistent with the data, to which robust control techniques are applied. The case that the energy of the process noise is bounded has also been studied in [18, 44, 169]. The paper [44] uses Young's inequality to develop sufficient conditions for data-driven stabilization. Additional sufficient conditions for data-driven stabilization are provided in [18], based on linear fractional transformations. The first necessary and sufficient conditions for data-driven stabilization of linear systems were developed in [169], based on a matrix version of the S-lemma (see also Chapter A). These results were further extended in [23] using the related Petersen's lemma, and a unification of these results is provided in [168]. Furthermore, data-driven stabilization using measurement noise satisfying an energy bound has been studied in [24]. The works [121, 122] take a different approach and quantify uncertainty using a distance between finite-dimensional subspaces containing restricted trajectories of the system. In addition to these contributions, also statistical assumptions on the noise have been considered, see for example the paper [162] that addresses the LQR problem in the setting of Gaussian noise, [197] that introduces a maximum likelihood framework with applications to predictive control, and [124] that studies stochastic optimal control from the perspective of the fundamental lemma.

Yet another line of research involves the extension of data-driven control techniques for discrete-time linear time-invariant dynamics to other system classes. For example, data-driven control of continuous-time systems has been studied in [21, 49, 136, 145]. We also point out methods for data-driven (absolute) stabilization of Lur'e systems [99, 167, 168], i.e., systems that are the feedback interconnection of a linear system and a static nonlinearity. In addition, data-driven control of polynomial systems [68, 74, 75, 106] and rational systems [158] has received attention. Here, the vector field of the system is assumed to be a linear combination of known polynomials or rational functions. In [4, 5], data-driven control of a class of feedback linearizable systems was considered. Here, the linearizing feedback is assumed to be (approximately) equal to a linear combination of known basis functions, and the purpose is to identify the coefficients of this linear combination using data. Positive linear systems have been tackled in [81, 111]. In addition, methods based on Koopman operator theory have been used in [56, 85, 93, 151], with the idea of lifting the nonlinear dynamics to an (infinite-dimensional) linear system. Finally, we mention contributions to data-driven control of linear time-varying [120], linear parameter-varying systems [112, 179], and networked control systems [3, 15, 83].

Data-driven stabilization is a prototypical control problem that has been studied in several papers, including some of the ones mentioned above. However, there are also many contributions that study additional performance guarantees such as linear quadratic regulation [44–46, 63, 76, 175], $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance [18, 20, 156, 169, 176], and model matching [28, 181]. In addition, data-driven tracking has been considered in [43, 161].

The interest in data-driven control has not only delivered a variety of new controllers for various classes of systems, but has also led to a revival of interest in the fundamental lemma. Its original proof was presented in the language of behavioral theory; an alternative proof for state-space systems was provided in [172]. The original fundamental lemma works with Hankel matrices of trajectories, but various other matrix structures have been considered such as mosaic-

Hankel matrices [172], Page matrices [37] and "trajectory matrices" [102]. The condition of persistency of excitation is sufficient to generate rich data that enable the parameterization of all system trajectories. However, it is in general not necessary. In [103], it was shown that for a class of controllable single-input systems, persistency of excitation is necessary and sufficient to generate such rich data for all possible initial conditions. Moreover, in [150] general multi-input systems were studied and it was shown that persistently exciting inputs precisely coincide with those inputs that lead to rich data for any initial condition and any controllable system. Generalizations to uncontrollable systems are presented in [102, 113, 198] and extensions to continuous-time systems in [97, 98, 133, 134]. Robust and quantitative versions are explored in [17, 38, 39] while frequency domain formulations have been considered in [52, 109]. Furthermore, the fundamental lemma has been generalized to various other model classes such as descriptor systems [146], flat nonlinear systems [4], linear parameter-varying systems [180], and stochastic ones [51].

2

The data informativity framework

In this chapter we will introduce the concept of data informativity. This notion will play a central role in this book. It will be shown to constitute a powerful framework that can be applied to a large number of data-driven system analysis, control design, and modelling problems.

2.1 Introduction

As was discussed in the previous chapter, in some situations data obtained from the physical system contain sufficient information to identify the true system model uniquely. For the situation that the data are noiseless it was explained in Subsection 1.2.2 that the true system can be identified from data provided that the input data are persistently exciting. An important role is played here by Willems' fundamental lemma as discussed in Subsection 1.2.1. It is not surprising that in such situations, analysis and control design can be based on the data directly. Indeed, Subsections 1.2.3 to 1.2.5 describe examples in which data-driven analysis and control problems are treated under the assumption that the input data are persistently exciting. In general however, it is not possible to uniquely identify the physical system because the input data may not be persistently exciting, may not contain a sufficient number of samples, or the data may be corrupted by noise.

Therefore, an intriguing question is the following: is it possible to verify system properties and/or to obtain controllers from data that do not contain sufficient information to uniquely identify the true system? In this book we will address this question. The answer will turn out to depend on the particular system property or control design problem at hand. For several properties and problems, we will show the answer to be affirmative. This is remarkable because it highlights situations in which direct data-driven control is more powerful than the indirect approach, i.e., the combination of unique system identification and model-based control. On the other hand, in some situations the answer is negative. Such a negative answer is also significant, because it reveals situations in which identifiability of the system is necessary for data-driven analysis/control.

In this book we will restrict ourselves to discrete-time, linear, time-invariant systems, both in state space form, and in the form of higher order autoregressive models. In that context, the informativity approach will be applied to establish data-based tests for verifying whether an unknown system satisfies certain system theoretic properties such as, for example, stability, stabilizability and controllability. If, for a certain system property, the data indeed contain sufficient information to verify that property then we will call the data *informative* for this system property. Data informativity will also turn out to provide direct methods to data driven control design in the context of several classical control problems, such as, for example, stabilization, the linear quadratic regulator problem, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control, and the problem of tracking and regulation. Before embarking on these concrete system analysis and control design problems, in the next section we will first introduce the framework of data informativity at a rather general level.

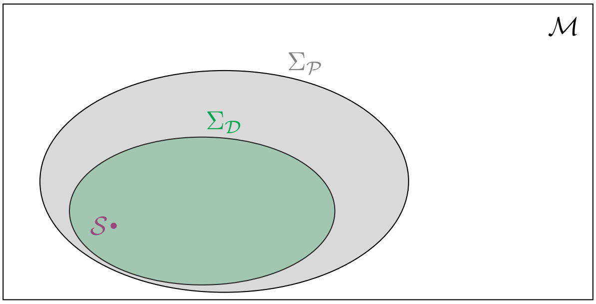
2.2 Data informativity

In this section we will introduce the concept of data informativity for verifying a given system property or solving a certain control design problem at a fairly abstract level. We will then illustrate this abstract setup by means of a series of concrete data driven analysis and control design problems.

To start with, we fix a certain model class $\mathcal{M}$. This model class is a given set of systems that is assumed to contain the 'true' system (i.e., a mathematical model of the underlying unknown physical system), denoted by $\mathcal{S}$. We assume that the true system $\mathcal{S}$ is not known but we do have access to a set of data, $\mathcal{D}$, generated by this system. More concretely, we think of $\mathcal{D}$ as the set collecting the data from some input-output experiment applied to the system $\mathcal{S}$. As explained in the introduction to this chapter, we are interested in assessing system-theoretic properties of $\mathcal{S}$ and designing control laws for it from the data $\mathcal{D}$. Given the set of data $\mathcal{D}$, we define $\Sigma_\mathcal{D} \subseteq \mathcal{M}$ to be the set of all systems in the model class $\mathcal{M}$ that are consistent with the data $\mathcal{D}$, i.e., that could also have generated the same data. In other words, it is impossible to distinguish the true system $\mathcal{S}$ from any other system in $\Sigma_\mathcal{D}$ on the basis of the given data $\mathcal{D}$ alone. This will be explained in more detail in several examples below.

We will first focus on data-driven analysis of system theoretic properties. Let $\mathcal{P}$ be some system theoretic property. We will denote the set of all systems within $\mathcal{M}$ having this property by $\Sigma_\mathcal{P}$. Suppose we are interested in the question whether our true system $\mathcal{S}$ has the property $\mathcal{P}$. Since the only information we have to base our answer on are the data $\mathcal{D}$ obtained from the true system, we can only conclude from the data that the true system has property $\mathcal{P}$ if *all* systems consistent with the data $\mathcal{D}$ have the property $\mathcal{P}$. If this is the case, we call the data informative for the system property. More precisely, this leads to the following definition, see also Figures 2.1 and 2.2.

Definition 2.1 (Informativity for system analysis)**.** We say that the data $\mathcal{D}$ are *informative* for property $\mathcal{P}$ if $\Sigma_{\mathcal{D}} \subseteq \Sigma_{\mathcal{P}}$, i.e., all systems that are consistent with the data have the property.



$\mathcal{M}$: model class
$\mathcal{S}$: unknown system
$\mathcal{D}$: given data set
$\Sigma_{\mathcal{D}}$: data consistent systems
$\mathcal{P}$: system property
$\Sigma_{\mathcal{P}}$: systems with property $\mathcal{P}$

Figure 2.1: The data are informative for property $\mathcal{P}$ as $\Sigma_{\mathcal{D}} \subseteq \Sigma_{\mathcal{P}}$.

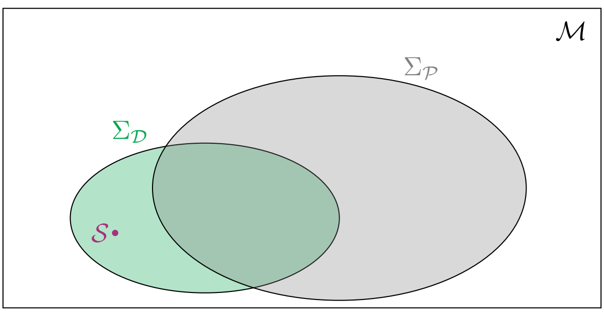


Figure 2.2: The data are not informative for property $\mathcal{P}$. Depending on the situation, either $\mathcal{S} \in \Sigma_{\mathcal{P}}$ or $\mathcal{S} \notin \Sigma_{\mathcal{P}}$. On the basis of the given data $\mathcal{D}$, it is impossible to distinguish these two cases.

Example 2.2. For given n and m, let the model class $\mathcal{M}$ be the set of all linear input-state systems of the form

$$x(t+1) = Ax(t) + Bu(t)$$

where x is the n-dimensional state and u is the m-dimensional input. Let the true system $\mathcal{S}$ be represented by the matrices $(A_{\text{true}}, B_{\text{true}})$.

An example of a data set $\mathcal{D}$ arises when considering data-driven problems on the basis of input and state measurements. Suppose that we collect input and state samples $u(0), u(1), \dots, u(T-1)$ and $x(0), x(1), \dots, x(T)$ on the time interval $[0, T]$. Recall the notation

$$U_{[0,T-1]} = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix}$$
$$X_{[0,T]} = \begin{bmatrix} x(0) & x(1) & \cdots & x(T) \end{bmatrix}$$

from Chapter 1. Here, in addition, we will use the shorthand notation

$$U_- := U_{[0,T-1]} \tag{2.1a}$$
$$X := X_{[0,T]}. \tag{2.1b}$$

By defining

$$X_- := X_{[0,T-1]} \tag{2.2a}$$
$$X_+ := X_{[1,T]} \tag{2.2b}$$

we clearly have $X_+ = A_{\text{true}} X_- + B_{\text{true}} U_-$ because the true system is assumed to generate the data.

We then define the data as $\mathcal{D} := (U_-, X)$. In this case, the set $\Sigma_{\mathcal{D}}$ is equal to $\Sigma_{(U_-,X)}$ defined by

$$\Sigma_{(U_-,X)} := \left\{ (A, B) \in \mathcal{M} \mid X_+ = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\}. \tag{2.3}$$

Clearly, we have $(A_{\text{true}}, B_{\text{true}}) \in \Sigma_{\mathcal{D}}$.

Suppose that we are interested in the system theoretic property $\mathcal{P}$ of *stabilizability*. The corresponding set $\Sigma_{\mathcal{P}}$ is then equal to Σ_{stab} defined by

$$\Sigma_{\text{stab}} := \{(A, B) \in \mathcal{M} \mid (A, B) \text{ is stabilizable}\}.$$

Then, the data (U_-, X) are informative for stabilizability if $\Sigma_{(U_-,X)} \subseteq \Sigma_{\text{stab}}$, that is, if all systems consistent with the input-state measurements are stabilizable. ■

In general, if the true system $\mathcal{S}$ can be uniquely determined from the data $\mathcal{D}$, that is $\Sigma_{\mathcal{D}} = \{\mathcal{S}\}$ and $\mathcal{S}$ has the property $\mathcal{P}$, then it is evident that the data $\mathcal{D}$ are informative for $\mathcal{P}$. However, the converse may not be true: $\Sigma_{\mathcal{D}}$ might contain many systems, all of which have property $\mathcal{P}$. In this book, we are interested in necessary *and* sufficient conditions for informativity of the data. Such conditions reveal the minimal amount of information required to assess the property $\mathcal{P}$. A natural problem statement is therefore the following:

Problem 2.3 (Informativity for system analysis)**.** Provide necessary and sufficient conditions on the data $\mathcal{D}$ under which these data are informative for property $\mathcal{P}$.

The above gives us a general framework to deal with data-driven analysis problems. Such analysis problems will be one of the main subjects of this book.

We will also deal with data-driven control problems. The objective in such problems is the data-based design of controllers such that the closed loop system, obtained from the interconnection of the true system $\mathcal{S}$ and the controller, satisfies the given control objective. As for the analysis problem, we have only the information from the data to base our design on. Therefore, we can only guarantee that our control objective is achieved if the designed controller achieves the design objective when interconnected with *any* system from the set $\Sigma_{\mathcal{D}}$.

For the framework to allow for data-driven control problems, we will consider a given control objective $\mathcal{O}$ (for example, a system theoretic property or a guaranteed performance of the closed loop system). Denote by $\Sigma_{\mathcal{O}}$ the set of all systems that satisfy the control objective $\mathcal{O}$. For a given controller $\mathcal{K}$, denote by $\Sigma_{\mathcal{D}}(\mathcal{K})$ the set of all systems obtained as the interconnection of a system in $\Sigma_{\mathcal{D}}$ with the controller $\mathcal{K}$. We then have the following variant of informativity:

Definition 2.4 (Informativity for control)**.** We say that the data $\mathcal{D}$ are *informative* for the control objective $\mathcal{O}$ if there exists a controller $\mathcal{K}$ such that $\Sigma_{\mathcal{D}}(\mathcal{K}) \subseteq \Sigma_{\mathcal{O}}$.

Example 2.5. In order to illustrate the above formal definition, in this example we will consider data driven stabilization by state feedback. In that context, for systems and data like in Example 2.2, we take the control objective $\mathcal{O}$: 'interconnection with a state feedback controller yields a stable closed loop system'. The set of all systems that satisfy the control objective is then equal to

$$\Sigma_{\mathcal{O}} = \{A \in \mathbb{R}^{n\times n} \mid A \text{ is stable}^{1}\}.$$

For a given state feedback controller $\mathcal{K} = K \in \mathbb{R}^{m\times n}$, the corresponding set of closed loop systems consistent with the data is equal to

$$\Sigma_{\mathcal{D}}(\mathcal{K}) = \{A + BK \mid (A, B) \in \Sigma_{\mathcal{D}}\}.$$

The data $\mathcal{D}$ are thus informative for the control objective $\mathcal{O}$ if there exists a single controller K such that $A + BK$ is stable for all $(A, B) \in \Sigma_{\mathcal{D}}$. ■

Obviously, the first step in any data-driven control problem is to determine whether it is possible to obtain, from the given data, a suitable controller. This leads to the following informativity problem:

[1] We say that a matrix is *stable* if all its eigenvalues are contained in the open unit disk.

Problem 2.6 (Informativity for control)**.** Provide necessary and sufficient conditions on $\mathcal{D}$ under which the data are informative for the control objective $\mathcal{O}$.

The second step of data-driven control involves the design of a suitable controller. In terms of our framework, this can be stated as:

Problem 2.7 (Data-driven control design)**.** Under the assumption that the data $\mathcal{D}$ are informative for the control objective $\mathcal{O}$, find a controller $\mathcal{K}$ such that $\Sigma_{\mathcal{D}}(\mathcal{K}) \subseteq \Sigma_{\mathcal{O}}$.

Example 2.8. In our previous example, we have considered stabilization using data obtained from input and state measurements. In the present example we will illustrate that also output measurements can be taken into account, and consider data driven stabilization by dynamic output feedback. In that context, our model class $\mathcal{M}$ consists of all systems of the form

$$x(t+1) = Ax(t) + Bu(t) \tag{2.4a}$$

$$y(t) = Cx(t) + Du(t). \tag{2.4b}$$

Here, x is the n-dimensional state, u is the m-dimensional input and y is the p-dimensional output. The dimensions n, m and p are given, fixed, integers. The unknown, true system $\mathcal{S}$ is given by the matrices $A_{\text{true}}, B_{\text{true}}, C_{\text{true}}$, and D_{true}.

Suppose that we have collected input-state-output data on the time interval $[0, T]$. Let U_-, X, X_-, and X_+ be defined by (2.1) and (2.2) and let Y_- be defined in a similar way as U_- by $Y_- := Y_{[0,T-1]}$. Our data are given by $\mathcal{D} = (U_-, X, Y_-)$. Since these data are assumed to be generated by the true system $(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})$ we have

$$\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A_{\text{true}} & B_{\text{true}} \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}.$$

The set of all systems that are consistent with these data is then given by:

$$\Sigma_{\mathcal{D}} = \left\{ (A, B, C, D) \mid \begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\}.$$

We want to design a stabilizing dynamic controller $\mathcal{K}$ of the form

$$z(t+1) = Kz(t) + Ly(t) \tag{2.5a}$$

$$u(t) = Mz(t). \tag{2.5b}$$

Here, the controller state z is q-dimensional, where the controller dimension q needs to be designed as well.

The design objective $\mathcal{O}$ is: 'interconnection with a dynamic controller yields a stable closed loop system'. Obviously, the set of all systems satisfying the control objective is

$$\Sigma_{\mathcal{O}} = \{A' \in \mathbb{R}^{(n+q)\times(n+q)} \mid q \in \mathbb{N}, \text{ and } A' \text{ is stable}\}.^{2}$$

For a given dynamic controller $\mathcal{K}$ of the form (2.5), the corresponding set of closed loop systems consistent with the data is equal to

$$\Sigma_{\mathcal{D}}(\mathcal{K}) = \left\{ \begin{bmatrix} A & BM \\ LC & K + LDM \end{bmatrix} \mid (A, B, C, D) \in \Sigma_{\mathcal{D}} \right\}.$$

The data $\mathcal{D}$ are informative for stabilization by dynamic output feedback if there exists a single controller $\mathcal{K}$ such that $\Sigma_{\mathcal{D}}(\mathcal{K}) \subseteq \Sigma_{\mathcal{O}}$, i.e. the controller $\mathcal{K}$ stabilizes all systems in $\mathcal{M}$ that are consistent with the data $\mathcal{D}$. The problem is to find necessary and sufficient under which $\mathcal{D}$ satisfies this property and, if so, to design a suitable controller $\mathcal{K}$. ■

Example 2.9. In some situations state data cannot be obtained, and only input and output data are available. This means that our data are of the form $\mathcal{D} = (U_-, Y_-)$, where

$$\begin{aligned} U_- &= U_{[0,T-1]} \\ Y_- &= Y_{[0,T-1]}. \end{aligned}$$

Assuming that our model class is still given by all systems of the form (2.4) with dimensions n, m, and p given, the set of all system consistent with the data becomes

$$\Sigma_{\mathcal{D}} := \left\{ (A, B, C, D) \mid \exists X \in \mathbb{R}^{n\times(T+1)} \text{ s.t. } \begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\}.$$

Again, the data $\mathcal{D}$ are informative for stabilization by dynamic output feedback if there exists a single controller $\mathcal{K}$ such that $\Sigma_{\mathcal{D}}(\mathcal{K}) \subseteq \Sigma_{\mathcal{O}}$. If this holds, the problem is to design a suitable dynamic controller $\mathcal{K}$. ■

As stated in the introduction, in this book we will highlight the strength of the informativity framework by solving multiple problems. In addition to the noise-free setting of the previous examples, we will also consider data driven analysis and control problems where the model class $\mathcal{M}$ consists of system models with unknown process noise and/or measurement noise. An example of this is the problem of quadratic stabilization by state feedback as illustrated below.

[2] We denote by $\mathbb{N} = \{1, 2, ...\}$ the set of natural numbers.

Example 2.10. For given n and m, consider the model class $\mathcal{M}$ consisting of all discrete-time linear input-state systems with unknown process noise of the form

$$x(t+1) = Ax(t) + Bu(t) + w(t)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input and $w(t) \in \mathbb{R}^n$ is an unknown noise term. Suppose we have data (U_-, X) given by (2.1). The noise w is unknown, so $w(0), w(1), \ldots, w(T-1)$ are not measured and hence are not part of the data. However, as part of the data $\mathcal{D}$ we do assume that we have the following information on the noise during the data sampling period: the individual noise samples $w(0), w(1), \ldots, w(T-1)$ satisfy the pointwise norm bound

$$\|w(t)\|_2^2 \leqslant \varepsilon \quad \text{for} \quad t \in [0, T-1] \tag{2.7}$$

for some known upper bound $\varepsilon > 0$. In other words, the data $\mathcal{D}$ consist of the measurements (U_-, X) together with the information that the noise on the sampling interval satisfies the inequality (2.7). Define

$$W_- := W_{[0,T-1]}.$$

Then we see that the set $\Sigma_{\mathcal{D}}$ is equal to the set of all systems (A, B) explaining the numerical data (U_-, X) together with the information (2.7) on the noise, i.e., all (A, B) satisfying

$$X_+ = AX_- + BU_- + W_- \tag{2.8}$$

for some W_- satisfying (2.7):

$$\Sigma_{\mathcal{D}} = \{(A, B) \mid (2.8) \text{ holds for some } W_- \text{ satisfying } (2.7)\}.$$

We will now formulate a control objective $\mathcal{O}$. Let Q be a given real positive definite $n \times n$ matrix. The control objective $\mathcal{O}$ depends on this given matrix Q and is taken as: 'interconnection with a state feedback controller yields a stable closed loop system with Lyapunov function $V(x) = x^\top Qx$ '. Hence the set $\Sigma_{\mathcal{O}}$ is given by

$$\Sigma_{\mathcal{O}} = \{A \in \mathbb{R}^{n \times n} \mid Q - A^\top QA > 0\}.$$

For a given state feedback controller $\mathcal{K} = K \in \mathbb{R}^{m \times n}$, the set of closed loop systems is equal to

$$\Sigma_{\mathcal{D}}(\mathcal{K}) = \{A + BK \mid (A, B) \in \Sigma_{\mathcal{D}}\}.$$

In accordance with Definition 2.4, the data $\mathcal{D}$ are informative for the control objective $\mathcal{O}$ if there exists a single controller K such that $A + BK$ is stable with Lyapunov function $V(x) = x^\top Qx$ for all $(A, B) \in \Sigma_{\mathcal{D}}$.

The informativity problem is now to find necessary and sufficient conditions on the data $\mathcal{D}$ (so on the numerical data (X, U_-) and on the positive real number ε that represent the bound on the noise) under which there exists a positive definite matrix Q such that the data are informative for the control objective $\mathcal{O}$. In that case, we will call the data informative for quadratic stabilization. In addition, we want to find a suitable controller $\mathcal{K}$. ■

In this book, we will extensively study a range of analysis and control design problems within the data informativity framework, including those illustrated in the examples of this chapter. As explained before, the main tool for verifying whether a given system property holds for the unknown system will be to check whether the property holds for *all systems in the model class that are consistent with the data*. Any test for verifying this will always be a test in terms of the given data set, and such a test will obviously depend on the particular property that needs to be verified. Once all systems in the set of systems that are consistent with the data satisfy the given system property, also the true system will.

In the same way, the crucial idea in designing a controller that achieves a given control objective for the unknown system will be to design a single controller that achieves the design objective for *all* systems consistent with the data. Here, the main problem will be to establish necessary and sufficient conditions for the existence of such controller. These conditions will be in the form of a test on the data. Once this test confirms that a suitable controller indeed exists, the problem is to design such controller based on the given data. In the end, this controller will then achieve the control objective for all systems consistent with the data.

Note that this setup is reminiscent of the problem of robust control design. In fact, in robust control one typically considers a nominal system with an uncertainty set around it. There, the problem is to find controllers that achieve the design objective for all systems in the uncertainty set. The framework that we consider in this book circumvents finding a nominal system and an uncertainty set. Instead, we work directly with the set of consistent systems induced by the given data. In other words, the 'system uncertainty' is determined immediately by the given data, and no attempt is made to find a nominal system and an uncertainty description that is suitable for existing methods in robust control design. The given data are called informative for a given design objective if the associated robust control problem allows a solution for the system uncertainty imposed by the data.

Of course, once a model class is given and data have been obtained, the set of systems consistent with these data will be nonempty, since as a standing assumption we assume that the unknown true system has generated these data, and is therefore consistent with the data. In general, this set will contain an infinite number of systems: all systems that could also have generated the same data. We will illustrate this by means of the following example.

Example 2.11. Consider the true (but unknown) system

$$x(t+1) = A_{\text{true}}x(t) + B_{\text{true}}u(t)$$

where A_{true} and B_{true} are given by

$$A_{\text{true}} = \begin{bmatrix} 1.5 & 0 \\ 1 & 0.5 \end{bmatrix}, \quad B_{\text{true}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We collect data from this system on the time interval from $t = 0$ until $t = 2$, which results in the data $\mathcal{D} = (X, U_-)$ with

$$X = \begin{bmatrix} 1 & 0.5 & -0.25 \\ 0 & 1 & 1 \end{bmatrix}, \quad U_- = \begin{bmatrix} -1 & -1 \end{bmatrix}.$$

As model class $\mathcal{M}$ we take

$$\mathcal{M} = \{(A, B) \mid A \in \mathbb{R}^{2\times 2}, B \in \mathbb{R}^{2\times 1}\}.$$

Recall from Example 2.2 that the set of all systems $\Sigma_{\mathcal{D}}$ that are consistent with the data is given by

$$\Sigma_{(U_-,X)} = \left\{(A, B) \in \mathcal{M} \mid X_+ = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}\right\}.$$

It is easily verified that $(A, B) \in \Sigma_{(U_-,X)}$ if and only if

$$A = \begin{bmatrix} 1.5 + a_1 & 0.5a_1 \\ 1 + a_2 & 0.5 + 0.5a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 + a_1 \\ a_2 \end{bmatrix}$$

for some $a_1, a_2 \in \mathbb{R}$. Thus, $\Sigma_{(U_-,X)}$ is an (infinite) affine subset of $\mathbb{R}^{2\times 2} \times \mathbb{R}^{2\times 1}$. In fact, it will be shown in Chapter 6 of this book that the data given above are informative for stabilization by state feedback, meaning that a single state feedback control law $u = Kx$ exists that stabilizes all systems in $\Sigma_{(U_-,X)}$, so also the unknown, true system. ■

A particular special case occurs if, for a given model class and a given set of data, the set of systems consistent with these data is a *singleton*, i.e. consists of exactly one system. If this is the case, then this single system must be the true system. We will then call the data *informative for system identification.*

Definition 2.12. Let $\mathcal{M}$ be a model class and let $\mathcal{D}$ be a given set of data. We say that the data $\mathcal{D}$ are informative for system identification if the set $\Sigma_{\mathcal{D}}$ contains exactly one element.

The following is then a natural, high level, formulation of the problem of system identification.

Problem 2.13 (System identification)**.** Find necessary and sufficient conditions on the data $\mathcal{D}$ to be informative for system identification. If so, determine the unique element of $\Sigma_{\mathcal{D}}$.

In case the data are informative for identification, then verifying a system property within the informativity framework amounts to verifying this property for the unique system consistent with the data (i.e., the true system). On a conceptual level, this can be done using two different approaches. A first approach is to actually identify the true system using the data, and subsequently verify the given property using existing (model based) methods. A second approach is to directly verify the given system property using a test in terms of the data. As explained before in this introduction, the first approach is called *indirect*, whereas the second approach is called *direct*. Similarly, in case the data are informative for identification, we can distinguish between the direct and the indirect approach to control design.

We will illustrate the distinction between the indirect and direct approach by means of the following example.

Example 2.14. For given n and m, let the model class $\mathcal{M}$ be the set of all linear input-state systems of the form

$$x(t+1) = Ax(t) + Bu(t).$$

As in Example 2.2, suppose we have collected data (U_-, X). The set $\Sigma_{\mathcal{D}}$ is equal to $\Sigma_{(U_-,X)}$ defined by (2.3). It will be shown later in this book (see Section 3.1) that the data $\mathcal{D}$ are informative for system identification if and only if the $(n+m) \times T$ matrix

$$\begin{bmatrix} X_- \\ U_- \end{bmatrix} \tag{2.9}$$

has full row rank (i.e., has linearly independent rows). If this is the case, then the unique element $(A_{\text{true}}, B_{\text{true}})$ of $\Sigma_{\mathcal{D}}$ (which must be the true system that has actually generated the data) is given by

$$\begin{bmatrix} A_{\text{true}} & B_{\text{true}} \end{bmatrix} = X_+ \begin{bmatrix} V_1 & V_2 \end{bmatrix} \tag{2.10}$$

where $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ is any right inverse[3] of the matrix in (2.9).

Suppose now that we want to check internal stability of our unknown system. Call this property $\mathcal{P}$. The indirect approach (assuming that we have informativity for identification) for verifying the property $\mathcal{P}$ is now to compute A_{true}

[3]We say that V is a right inverse of a matrix M if $MV = I$. We denote any such right inverse by $M^\sharp$.

using (2.10) and to apply any standard test for checking whether A_{true} is stable (i.e., has all its eigenvalues inside the unit disc).

Instead, a direct approach is to establish a test *in terms of the data directly*, without the intermediate step of identification. Indeed, later on in this book (see Section 3.3) it will be shown that *all* system in $\Sigma_{\mathcal{D}}$ have the property $\mathcal{P}$ if and only if the data $\mathcal{D}$ satisfy the following conditions: X_- has full row rank and it has a right inverse $X_-^\sharp$ such that $U_-X_-^\sharp = 0$ and $X_+X_-^\sharp$ is stable. Clearly, this test circumvents the identification step and is completely in terms of the data. It should be noted that the latter test does not even require informativity for identification. ∎

In the previous example, we have discussed informativity for system identification in the presence of input-state data. A more challenging problem is identification in the situation where the state of the system is not measured and its dimension is not given. This is discussed in the following example.

Example 2.15. Given integers $m, p \in \mathbb{N}$ and $N \in \mathbb{Z}_+$, consider the controllable and observable input-state-output system

$$\begin{aligned} x(t+1) &= A_{\text{true}}x(t) + B_{\text{true}}u(t) \\ y(t) &= C_{\text{true}}x(t) + D_{\text{true}}u(t) \end{aligned}$$

where $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, and $x(t) \in \mathbb{R}^{n_{\text{true}}}$ with $n_{\text{true}} \leqslant N$. Here, the true system matrices $A_{\text{true}}, B_{\text{true}}, C_{\text{true}}$ and D_{true} are unknown. Also, the true state-space dimension n_{true} is unknown, but an upper bound N is given. In this setting, the model class $\mathcal{M}$ consists of all controllable and observable systems

$$(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{m\times p}$$

with m inputs, p outputs, and $n \leqslant N$ states. We collect input-output data (U_-, Y_-) from the true system. Now, a system $(A, B, C, D) \in \mathcal{M}$ is *consistent* with the input-output data if

$$\begin{aligned} X_+ &= AX_- + BU_- \\ Y_- &= CX_- + DU_- \end{aligned} \tag{2.11}$$

holds for some $X \in \mathbb{R}^{n\times(T+1)}$ with $n \leqslant N$. The set of all systems consistent with the data (U_-, Y_-) is denoted by $\Sigma_{(U_-,Y_-)}$. We call the data (U_-, Y_-) *informative for system identification* if all systems in $\Sigma_{(U_-,Y_-)}$ have precisely n_{true} states, and any pair of systems in $\Sigma_{(U_-,Y_-)}$ is isomorphic, that is,

$$(A, B, C, D), (\bar{A}, \bar{B}, \bar{C}, \bar{D}) \in \Sigma_{(U_-,Y_-)}$$

implies that there exists a nonsingular $S \in \mathbb{R}^{n_{\text{true}}\times n_{\text{true}}}$ such that

$$SAS^{-1} = \bar{A}, \ \ SB = \bar{B}, \ \ CS^{-1} = \bar{C}, \quad \text{and } D = \bar{D}.$$

In other words, the data (U_-, Y_-) uniquely determine the true state-space dimension n_{true}, and the true system matrices $A_{\text{true}}, B_{\text{true}}, C_{\text{true}}$ and D_{true} up to a similarity transformation. In Chapter 11 we will provide necessary and sufficient conditions on the input-output data to be informative for system identification. ■

If our model class $\mathcal{M}$ involves systems with process and/or measurement noise, then in general a set of data $\mathcal{D}$ will not be informative for identification, i.e. the set $\Sigma_{\mathcal{D}}$ will contain infinitely many systems. This can, for example, be seen from Example 2.10, where $\Sigma_{\mathcal{D}}$ will contain infinitely many elements, regardless of the algebraic properties of the numerical data (U_-, X), due to the presence of the wide range of noise matrices W_- satisfying the inequality (2.7). This observation highlights the fact that the informativity framework, as it is dedicated to the direct approach to system analysis and control design, can be particularly useful in the context of systems with noise.

Although in this book we will be mainly concerned with the direct approach, in the context of systems without noise a natural question is how to generate data that are informative for identification. This question touches upon the problem of *experiment design*, an issue that will be addressed in Chapter 12 of this book. This problem is formulated as follows.

Problem 2.16 (Experiment design)**.** Given a model class $\mathcal{M}$, design an experiment to generate data $\mathcal{D}$ that are informative for system identification.

We will illustrate the issue of experiment design in the following example.

Example 2.17. For given n and m, consider the model class $\mathcal{M}$ consisting of all controllable linear input-state systems of the form

$$x(t+1) = Ax(t) + Bu(t)$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. Our aim is to obtain data (U_-, X) that are informative for system identification, equivalently, the matrix in (2.9) has full row rank. First, note that this matrix is $(n+m) \times T$, where T represents the length of the sampling interval. Suppose now that we are free to choose T and a suitable input sequence $u_{[0,T-1]}$. Choosing this input sequence in an appropriate way will be our experiment design. It follows immediately from the fundamental lemma (see Theorem 1.2) that if we choose the input sequence $u_{[0,T-1]}$ to be persistently exciting of order $n+1$, i.e., the Hankel matrix

$$H_{n+1}(u_{[0,T-1]}) = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-n-1) \\ u(1) & u(2) & \cdots & u(T-n) \\ \vdots & \vdots & & \vdots \\ u(n) & u(n+1) & \cdots & u(T-1) \end{bmatrix} \tag{2.12}$$

has full row rank, then indeed for any initial state $x(0)$ the resulting data matrix (2.9) has full row rank. Thus, within the model class of controllable input-state systems one can always uniquely identify the true system by first applying to the unknown system a suitable (persistently exciting) finite length input sequence, followed by 'harvesting' a corresponding state sequence. ■

We will conclude this chapter with an example illustrating data informativity for systems with *unbounded* noise.

Example 2.18. Consider the linear input-state-output systems with noise given by

$$x(t+1) = A_{\text{true}}x(t) + Bu(t) + Ew(t) \tag{2.13a}$$
$$y(t) = Cx(t) + Du(t) + Fw(t) \tag{2.13b}$$

where u is the m-dimensional control input, x is the n-dimensional state, y the p-dimensional output, and w r-dimensional unknown noise. In this example, we assume that the system is only partly unknown, in the sense that the true state matrix A_{true} is unknown and can be any real $n \times n$ matrix, but the matrices B, C, D, E and F are known. The term Ew represents process noise, whereas Fw represents measurement noise. Thus, our model class $\mathcal{M}$ consists of all systems of the form

$$x(t+1) = Ax(t) + Bu(t) + Ew(t) \tag{2.14a}$$
$$y(t) = Cx(t) + Du(t) + Fw(t) \tag{2.14b}$$

parametrized by $A \in \mathbb{R}^{n\times n}$.

We assume that we have input-state-output data $\mathcal{D} = (U_-, X, Y_-)$ on a given finite time interval $[0, T]$. A system in $\mathcal{M}$ is consistent with the data if and only if there exists a real $r \times T$ matrix W_- such that

$$X_+ = AX_- + BU_- + EW_- \tag{2.15a}$$
$$Y_- = CX_- + DU_- + FW_-. \tag{2.15b}$$

Hence the set of systems in $\mathcal{M}$ consistent with the data can be represented as

$$\Sigma_{\mathcal{D}} = \{A \in \mathbb{R}^{n\times n} \mid \text{ there exists } W_- \text{ such that (2.15) holds}\}$$

As an example, take the property $\mathcal{P}$ as: 'the system is detectable', i.e., the 'true' pair (C, A_{true}) is a detectable pair. Then, as before, the data $\mathcal{D}$ are informative for property $\mathcal{P}$ if for all $A \in \Sigma_{\mathcal{D}}$ the pair (C, A) is detectable. The problem is to find necessary and sufficient conditions on $\mathcal{D}$ for this to hold. Yet another property could be: 'the system is controllable', i.e. the true pair (A_{true}, B) is a controllable pair. The data are informative for this property if for all $A \in \Sigma_{\mathcal{D}}$ the pair (A, B) is controllable. These issues will be discussed in detail in Chapter 5 of this book. ■

2.3 Notes and references

In this chapter, we have defined a general notion of data informativity for *system analysis and control design*. The first paper that studied data informativity in this context was [175], see also the overview paper [173]. The terminology of ‘data informativity’ finds its roots in system identification [60, 61, 96], where informativity is usually understood as a condition on the data under which it is possible to distinguish between *different models* in a (parametric) model class. Here, we are not necessarily interested in distinguishing between different models, but rather want to understand whether it is possible to assess a system-theoretic property, or to synthesize a controller using the data. As we will see in later chapters of this book, this is often possible even when the data do not allow us to distinguish between different (data-consistent) systems.

Part I

Data-driven Analysis

3

System properties from data

This chapter deals with data-driven analysis of basic system-theoretic properties. We will establish tests on the data to be informative for identification, controllability, stability and stabilizability. This will be done both for systems with noise and for the noiseless case.

3.1 Informativity for identification

In this section, we will study informativity for identification as defined in Definition 2.12 and illustrated in Example 2.14. As in Example 2.2, suppose we have some unknown system $\mathcal{S}$ given by

$$x(t+1) = A_{\text{true}}x(t) + B_{\text{true}}u(t) \tag{3.1}$$

where x is the n-dimensional state and u is the m-dimensional input. The dimensions n and m are assumed to be known, but the matrices $(A_{\text{true}}, B_{\text{true}})$ are unknown. We embed the unknown system $\mathcal{S}$ into the model class $\mathcal{M}$, which we take as the set of all discrete-time linear input-state systems (with given state space dimension n and input dimension m) of the form

$$x(t+1) = Ax(t) + Bu(t). \tag{3.2}$$

Suppose the data set is given by $\mathcal{D} = (U_-, X)$ with U_- and X collecting the data obtained from the true system (3.1) on the time interval $[0, T]$, defined by (2.1) as

$$\begin{aligned} U_- &= U_{[0,T-1]} = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix} \\ X &= X_{[0,T]} = \begin{bmatrix} x(0) & x(1) & \cdots & x(T) \end{bmatrix}. \end{aligned}$$

The set of all systems in $\mathcal{M}$ consistent with these data is then equal to $\Sigma_{(U_-,X)}$ defined by

$$\Sigma_{(U_-,X)} := \left\{ (A,B) \in \mathcal{M} \mid X_+ = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\} \tag{3.3}$$

with X_- and X_+ defined by (2.2) as

$$\begin{aligned} X_- &= X_{[0,T-1]} \\ X_+ &= X_{[1,T]}. \end{aligned}$$

By assumption we have $(A_{\text{true}}, B_{\text{true}}) \in \Sigma_{(U_-,X)}$.

Note that the defining equation of (3.3) is a system of linear equations in the unknowns A and B. The solution space of the corresponding homogeneous equation is denoted by $\Sigma^{\text{hom}}_{(U_-,X)}$ and is equal to

$$\Sigma^{\text{hom}}_{(U_-,X)} := \left\{ (A_0, B_0) \mid 0 = \begin{bmatrix} A_0 & B_0 \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\}. \tag{3.5}$$

Now recall from Definition 2.12 that the data (U_-, X) are informative for system identification if the set $\Sigma_{(U_-,X)}$ contains exactly one system. If this is the case, this system is necessarily equal to the unknown system $(A_{\text{true}}, B_{\text{true}})$. The following theorem gives necessary and sufficient conditions for this to hold:

Theorem 3.1. *The data (U_-, X) are informative for system identification if and only if*

$$\operatorname{rank} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = n + m. \tag{3.6}$$

Now, suppose that (3.6) holds. Then for any right inverse of $\begin{bmatrix} X_- \\ U_- \end{bmatrix}$, partitioned as

$$\begin{bmatrix} X_- \\ U_- \end{bmatrix}^\sharp = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$

where $V_1 \in \mathbb{R}^{T \times n}$ and $V_2 \in \mathbb{R}^{T \times m}$, we have $A_{\text{true}} = X_+ V_1$ and $B_{\text{true}} = X_+ V_2$.

Proof. Obviously, $\Sigma_{(U_-,X)}$ contains exactly one element if and only if the solution set (3.5) of the homogeneous equation only contains $(0, 0)$. This is the case if and only if (3.6) holds. For any right inverse $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ then, the unique solution $(A_{\text{true}}, B_{\text{true}})$ of the inhomogeneous linear equation

$$X_+ = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}$$

is then given by $(A_{\text{true}}, B_{\text{true}}) = X_+ \begin{bmatrix} V_1 & V_2 \end{bmatrix}$. ☐

Note that this result confirms the claims made in Example 2.14.

3.2 Controllability and stabilizability from data

As we will show in this section, condition (3.6) is not necessary to perform data-driven analysis in general. Indeed, we will establish data-driven tests for verifying controllability and stabilizability that do not require the data to be informative for system identification.

Recall the well-known Hautus test for controllability: a system (A, B) is controllable if and only if

$$\operatorname{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \tag{3.7}$$

for all $\lambda \in \mathbb{C}$. For stabilizability, the Hautus test requires that (3.7) holds for all λ outside the open unit disc. We refer to [160, Thm. 3.13] for a discussion of Hautus tests for continuous-time linear systems. For discrete-time systems, the conditions are analogous, as explained here.

Now recall from Definition 2.1 the definition of informativity of data for a given system property. In the following we apply this setup to study informativity of input-state data as introduced in Section 2.2 to the properties $\mathcal{P}$ of controllability and stabilizability. In accordance with Definition 2.1 we have the following notions of informativity for controllability and stabilizability:

Definition 3.2. We say that the data (U_-, X) are *informative for controllability* if all systems in $\Sigma_{(U_-,X)}$ are controllable and *informative for stabilizability* if all systems in $\Sigma_{(U_-,X)}$ are stabilizable.

The following theorem gives necessary and sufficient conditions on the input-state data to be informative for these two properties. The result provides tests on the given data matrices.

Theorem 3.3 (Data-driven Hautus tests)**.** *The data (U_-, X) are informative for controllability if and only if*

$$\operatorname{rank}(X_+ - \lambda X_-) = n \;\; \textit{for all } \lambda \in \mathbb{C}. \tag{3.8}$$

Similarly, the data (U_-, X) are informative for stabilizability if and only if

$$\operatorname{rank}(X_+ - \lambda X_-) = n \;\; \textit{for all } \lambda \in \mathbb{C} \textit{ with } |\lambda| \geqslant 1. \tag{3.9}$$

Before proving the theorem, we will discuss some of its implications. We begin with computational issues.

Remark 3.4. Similar to the classical Hautus test, (3.8) and (3.9) can be verified by checking the rank for finitely many complex numbers λ. Indeed, (3.8) is equivalent to $\operatorname{rank}(X_+) = n$ and

$$\operatorname{rank}(X_+ - \lambda X_-) = n$$

for all $\lambda \neq 0$ with $\lambda^{-1} \in \sigma(X_- X_+^\sharp)$, where $X_+^\sharp$ is any right inverse of X_+. Here, we recall that $\sigma(M)$ denotes the spectrum, i.e. set of eigenvalues of the matrix M. In order to prove this, note that one direction of this equivalence is obvious. Conversely, assume that (3.8) does not hold. Then there exists $\lambda \in \mathbb{C}$

and $v \in \mathbb{C}^n$, $v \neq 0$ such that $v^*(X_+ - \lambda X_-) = 0$. Since X_+ has rank n, it has a right-inverse $X_+^\sharp$, from which we obtain $v^*(I - \lambda X_- X_+^\sharp) = 0$. Since $v \neq 0$ we have $\lambda \neq 0$ and therefore also $v^*(\lambda^{-1} I - X_- X_+^\sharp) = 0$. Hence λ^{-1} is an eigenvalue of $X_- X_+^\sharp$, which yields a contradiction.

Similarly, (3.9) is equivalent to $\operatorname{rank}(X_+ - X_-) = n$ and

$$\operatorname{rank}(X_+ - \lambda X_-) = n$$

for all $\lambda \neq 1$ with $(\lambda - 1)^{-1} \in \sigma(X_-(X_+ - X_-)^\sharp)$, where $(X_+ - X_-)^\sharp$ is any right inverse of $X_+ - X_-$. The proof of this equivalence is left to the reader.

As announced at the beginning of this section, there are situations in which we can conclude controllability or stabilizability from the data without being able to identify the true system uniquely. This is illustrated in the following example.

Example 3.5. Suppose that $n = 2$ and $m = 1$. Assume we collect data on the time interval $[0, T]$ with $T = 2$ to obtain

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } U_- = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

This implies that

$$X_+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } X_- = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Clearly, by Theorem 3.3 we see that these data are informative for controllability, as

$$\operatorname{rank} \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix} = 2 \quad \forall \lambda \in \mathbb{C}.$$

Since therefore all systems consistent with the data are controllable, we conclude that the true system is controllable. Note that the data are not informative for system identification, because

$$\Sigma_{(U_-, X)} = \left\{ \left(\begin{bmatrix} 0 & a_1 \\ 1 & a_2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \mid a_1, a_2 \in \mathbb{R} \right\} \tag{3.10}$$

so the set of systems that are consistent with the data contains infinitely many elements. ■

Proof of Theorem 3.3. We will only prove the characterization of informativity for controllability. The proof for stabilizability uses very similar arguments, and is hence omitted.

Note that the condition (3.8) is equivalent to the implication:

$$z \in \mathbb{C}^n, \lambda \in \mathbb{C} \text{ and } z^* X_+ = \lambda z^* X_- \implies z = 0. \tag{3.11}$$

Suppose that the implication (3.11) holds. Let $(A, B) \in \Sigma_{(U_-,X)}$ and suppose that $z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix} = 0$. We want to prove that $z = 0$. Note that $z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix} = 0$ implies that

$$z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = 0$$

or equivalently $z^* X_+ = \lambda z^* X_-$. This means that $z = 0$ by (3.11). We conclude that (A, B) is controllable, i.e., the data (U_-, X) are informative for controllability.

Conversely, suppose that (U_-, X) are informative for controllability. Let $z \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ be such that $z^* X_+ = \lambda z^* X_-$. This implies that for all $(A, B) \in \Sigma_{(U_-,X)}$, we have $z^* \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = \lambda z^* X_-$. In other words,

$$z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = 0. \tag{3.12}$$

We now distinguish two cases, namely the case that λ is real, and the case that λ is complex. First suppose that λ is real. Without loss of generality, z is real. We want to prove that $z = 0$. Suppose on the contrary that $z \neq 0$ and $z^\top z = 1$. We define the (real) matrices

$$\bar{A} := A - zz^\top (A - \lambda I) \text{ and } \bar{B} := B - zz^\top B.$$

In view of (3.12), we find that $(\bar{A}, \bar{B}) \in \Sigma_{(U_-,X)}$. Moreover,

$$z^\top \bar{A} = z^\top A - z^\top (A - \lambda I) = \lambda z^\top$$

and

$$z^\top \bar{B} = z^\top B - z^\top B = 0.$$

This means that

$$z^\top \begin{bmatrix} \bar{A} - \lambda I & \bar{B} \end{bmatrix} = 0.$$

However, this is a contradiction as $(\bar{A}, \bar{B})$ is controllable by the hypothesis that the data (U_-, X) are informative for controllability. We conclude that $z = 0$ which shows that (3.11) holds for the case that λ is real.

Next, we consider the case that λ is complex, say $\lambda = \sigma + i\omega$ with $\omega \neq 0$. We write z as $z = p + iq$, where $p, q \in \mathbb{R}^n$. We now distinguish two special cases, the case that p and q are linearly dependent and, secondly, the case that they are linearly independent.

If p and q are linearly dependent, then $p = \alpha q$ or $q = \beta p$ for $\alpha, \beta \in \mathbb{R}$. If $p = \alpha q$ then substitution of $z = (\alpha + i)q$ into $z^* X_+ = \lambda z^* X_-$ yields

$$(\alpha - i)q^\top X_+ = (\sigma + i\omega)(\alpha - i)q^\top X_-.$$

Thus, $q^\top X_+ = (\sigma - \alpha\omega)q^\top X_-$, which means that $q = 0$ by the special case that λ is real treated before. Then also $p = 0$ so we find $z = 0$. Using the same arguments, we can show that $z = 0$ if $q = \beta p$.

Now assume that p and q are linearly independent. Since λ is complex, $n \geqslant 2$. Therefore, by linear independence of p and q there exist $\eta, \zeta \in \mathbb{R}^n$ such that

$$\begin{bmatrix} p^\top \\ q^\top \end{bmatrix} \begin{bmatrix} \eta & \zeta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We now define the real matrices $\bar{A}$ and $\bar{B}$ as

$$\begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} := \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} \eta & \zeta \end{bmatrix} \begin{bmatrix} \operatorname{Re}\left(z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix}\right) \\ \operatorname{Im}\left(z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix}\right) \end{bmatrix}.$$

By (3.12) we have $(\bar{A}, \bar{B}) \in \Sigma_{(U_-,X)}$. Next, we compute

$$\begin{aligned} z^* \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} &= z^* \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} \operatorname{Re}\left(z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix}\right) \\ \operatorname{Im}\left(z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix}\right) \end{bmatrix} \\ &= z^* \begin{bmatrix} A & B \end{bmatrix} - z^* \begin{bmatrix} A - \lambda I & B \end{bmatrix} \\ &= z^* \begin{bmatrix} \lambda I & 0 \end{bmatrix}. \end{aligned}$$

This implies that $z^* \begin{bmatrix} \bar{A} - \lambda I & \bar{B} \end{bmatrix} = 0$. Using the fact that $(\bar{A}, \bar{B})$ is controllable, we conclude that $z = 0$. This completes the proof of the theorem. □

3.3 Informativity for stability

In this section we will study how to check from the data obtained from some unknown system whether this system is stable. Again assume our model class $\mathcal{M}$ to consist of all systems of the form (3.2) with given state space dimension n and input dimension m. Our data set is given by $\mathcal{D} = (U_-, X)$ with U_- and X representing the data on the time interval $[0, T]$ as given by (2.1). As before, the set of all systems in $\mathcal{M}$ consistent with these data is given by (3.3).

Definition 3.6. We say that the data (U_-, X) are *informative for stability* if for all $(A, B) \in \Sigma_{(U_-,X)}$ the matrix A is stable, i.e. $|\lambda| < 1$ for all $\lambda \in \sigma(A)$.

The following theorem gives conditions on the data to be informative for stability.

Theorem 3.7. *The data* (U_-, X) *are informative for stability if and only if* X_- *has full row rank and there exists a right-inverse* $X_-^\sharp$ *of* X_- *such that* $X_+X_-^\sharp$ *is stable and* $U_-X_-^\sharp = 0$*. In that case we have that* $A = X_+X_-^\sharp$ *for all* $(A, B) \in \Sigma_{(U_-,X)}$*, so in particular* $A_{\text{true}} = X_+X_-^\sharp$*.*

Proof. To prove the 'if' statement, let $(A, B) \in \Sigma_{(U_-,X)}$. We need to prove that A is stable. We have $X_+ = AX_- + BU_-$. Assume that X_- has full row rank and take a right-inverse $X_-^\sharp$ with the properties as stated. Then clearly $X_+X_-^\sharp = AX_-X_-^\sharp + BU_-X_-^\sharp = A$, so A is stable.

To prove the 'only if' statement, let $z \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ be such that

$$\begin{bmatrix} z^\top & v^\top \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = 0.$$

Take a fixed $(A, B) \in \Sigma_{(U_-,X)}$. Then for all $\alpha \in \mathbb{R}$, the system $(A + \alpha zz^\top, B + \alpha vv^\top) \in \Sigma_{(U_-,X)}$ as well. Hence for all α, $A + \alpha zz^\top$ is stable. In particular this implies that $|\operatorname{tr}(A + \alpha zz^\top)| < n$. However, $\operatorname{tr}(A + \alpha zz^\top) = \operatorname{tr}(A) + \alpha z^\top z$ and therefore we must have $z = 0$. Thus,

$$\begin{bmatrix} z \\ v \end{bmatrix} \in \ker \begin{bmatrix} X_-^\top & U_-^\top \end{bmatrix} \text{ implies } z = 0$$

equivalently, $\ker \begin{bmatrix} X_-^\top & U_-^\top \end{bmatrix} \subseteq \ker \begin{bmatrix} I_n & 0 \end{bmatrix}$. By taking orthogonal complements, this yields

$$\operatorname{im} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} X_- \\ U_- \end{bmatrix}.$$

Thus there exists a matrix W such that

$$\begin{bmatrix} I_n \\ 0 \end{bmatrix} = \begin{bmatrix} X_- \\ U_- \end{bmatrix} W.$$

Define $X_-^\sharp := W$. This is a right-inverse of X_- with the property that $U_-X_-^\sharp = 0$. Moreover, $X_+X_-^\sharp = AX_-X_-^\sharp + BU_-X_-^\sharp = A$ is stable.

To prove the remaining statement, note that $(A, B) \in \Sigma_{(U_-,X)}$ implies that $X_+ = AX_- + BU_-$ so $X_+X_-^\sharp = AX_-X_-^\sharp + BU_-X_-^\sharp = A$. This completes the proof. □

The above theorem shows that if the data are informative for stability then they are informative for 'partial' identification in the sense that the true system matrix A_{true} is uniquely determined by the data. In general, however, the true input matrix B_{true} is not determined by the data. This is illustrated in the following example.

Example 3.8. Suppose our unknown system is $(A_{\text{true}}, B_{\text{true}})$ with

$$A_{\text{true}} = \begin{bmatrix} 0 & 0 \\ 1 & \frac{1}{2} \end{bmatrix} \text{ and } B_{\text{true}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Assume we have data for $t = 0, 1, 2$ given by

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}, \; U_- = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

Then $X_- = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $X_+X_-^{-1} = \begin{bmatrix} 0 & 0 \\ 1 & \frac{1}{2} \end{bmatrix}$ is stable, and $U_-X_-^{-1} = 0$. It can be shown that

$$\Sigma_{(U_-,X)} = \left\{ \left(\begin{bmatrix} 0 & 0 \\ 1 & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) \mid b_1, b_2 \in \mathbb{R} \right\}.$$

■

In the above, we have given necessary and sufficient conditions for informativity of input-state data for systems with inputs. A relevant issue that remains is to find conditions for informativity of data for *autonomous* systems, i.e. systems without inputs. In that case, the input matrix is absent, and the unknown system $\mathcal{S}$ is of the form

$$x(t+1) = A_{\text{true}}x(t) \tag{3.13}$$

where the state x is n-dimensional. This situation requires a different model class, namely the model class $\mathcal{M}_{\text{aut}}$ consisting of all autonomous systems

$$x(t+1) = Ax(t) \tag{3.14}$$

where A ranges over all real $n \times n$ matrices. As data we now have state measurements of the system $\mathcal{S}$, collected in the matrix X, again as defined in (2.1). The set of all autonomous systems consistent with these data is

$$\Sigma_X := \{A \mid X_+ = AX_-\}.$$

Definition 3.9. We call the state data X *informative for stability* if all matrices $A \in \Sigma_X$ are stable.

It turns out that the latter notion of informativity of state data is strongly related to informativity for *stabilizability* as studied in Section 3.2. In order to make this precise, recall the model class $\mathcal{M}$ of all input-state systems of the form (3.2) with given state space dimension n and input dimension m. We have the following lemma.

Lemma 3.10. *Let m be any positive integer. Then the state data X are informative for stability (in the sense of Definition 3.9) if and only if the input-state data $(0_{m,T}, X)$ are informative for stabilizability (in the sense of Definition 3.2).*

Proof. We first prove the 'only if' statement. Let $(A, B) \in \Sigma_{(0_{m,T}, X)}$, i.e. (A, B) is consistent with the input-state data $(0_{m,T}, X)$. We need to prove that (A, B) is stabilizable. Note that $X_+ = AX_- + B0_{m,T}$. Obviously, this implies that $X_+ = AX_-$. Since the state data X are informative for stability, this implies that A is stable. But then also (A, B) is stabilizable.

Next we prove the 'if' statement. Assume that the data $(0_{m,T}, X)$ are informative for stabilizability. Take any $A \in \Sigma_X$. Then $X_+ = AX_- = AX_- + 0_{n,m}0_{m,T}$, so $(A, 0_{n,m})$ is consistent with the input-state data $(0_{m,T}, X)$. This implies that the pair $(A, 0_{n,m})$ is stabilizable, hence A must be stable. □

Using this lemma, we are now able to characterize informativity of the state data.

Theorem 3.11. *The data X are informative for stability if and only if X_- has full row rank and $X_+X_-^\sharp$ is stable for any right inverse $X_-^\sharp$. In that case the set Σ_X contains exactly one element. This unique system is equal to A_{true} and $A_{\text{true}} = X_+X_-^\sharp$.*

Proof. We first prove the 'if' part. Assume that X_- has full row rank and $X_+X_-^\sharp$ is stable for any right inverse $X_-^\sharp$. Let A be in Σ_X. Since $X_+ = AX_-$, this immediately yields $A = X_+X_-^\sharp$, so A is stable. This proves that the data X are informative for stability.

Next we prove the 'only if' part. By Lemma 3.10, the (artificial) input-state data $(0_{m,T}, X)$ are informative for stabilizability, and hence it follows from Theorem 3.3 that

$$\operatorname{rank}(X_+ - \lambda X_-) = n \text{ for all } \lambda \in \mathbb{C} \text{ with } |\lambda| \geqslant 1. \tag{3.15}$$

Let z be such that $z^\top X_- = 0$. Take any $A \in \Sigma_X$ and λ such that $|\lambda| \geqslant 1$. Then λ is not an eigenvalue of A. Note that

$$z^\top (A - \lambda I)^{-1}(X_+ - \lambda X_-) = z^\top X_- = 0.$$

Since $\operatorname{rank}(X_+ - \lambda X_-) = n$, we may conclude that $z = 0$. Hence, X_- has full row rank. Therefore, the solution set of $X_+ = AX_-$, equivalently the set Σ_X, contains exactly one element. By informativity this element is stable, and is equal to $X_+X_-^\sharp$ for any right inverse $X_-^\sharp$, which is therefore stable. Finally, this unique element must be equal to A_{true}. □

Note from Theorem 3.3 that informativity for stability (with respect to the model class of autonomous systems) implies that the true system can be uniquely identified from the state data. This is in contrast with the notion of informativity for stability as defined in Definition 3.6 (defined with respect to the model class of input-state systems), where identifiability is not necessary, and where only the true system matrix A_{true} is uniquely determined by the data.

3.4 Systems with noise

So far, we have obtained data-driven tests for verifying stability, controllability and stabilizability of noiseless input-state systems. We have also established a test for stability of noiseless autonomous systems. In the remainder of this chapter we will study data-driven tests for input-state systems with noise and for autonomous systems with noise. Before embarking on this, in the present section we will first introduce the model classes of input-state systems with noise and autonomous systems with noise that we will be using, and discuss the assumptions that will be made on the noise samples.

We will start off with input-state systems. Suppose that the unknown, true system is given by

$$x(t+1) = A_{\text{true}}x(t) + B_{\text{true}}u(t) + w(t) \tag{3.16}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input and $w(t) \in \mathbb{R}^n$ is an unknown noise term. The matrices $A_{\text{true}} \in \mathbb{R}^{n\times n}$ and $B_{\text{true}} \in \mathbb{R}^{n\times m}$ denote the unknown state and input matrices. We embed this unknown system into the model class $\mathcal{M}$ of all input-state systems with unknown process noise, with fixed dimensions n and m, of the form

$$x(t+1) = Ax(t) + Bu(t) + w(t). \tag{3.17}$$

Suppose that we obtain data from the true system (3.16) on the time interval $[0,T]$. These data are given by (U_-, X). The noise w is unknown, so $w(0), w(1), \dots, w(T-1)$ are not measured, and therefore are not part of the data. In addition to the data $\mathcal{D}$ we do however assume that we have the following information on the noise during the data sampling period.

Assumption 3.12. The noise samples $w(0), w(1), \dots, w(T-1)$, collected in the matrix

$$W_- := W_{[0,T-1]}$$

satisfy the quadratic matrix inequality

$$\begin{bmatrix} I \\ W_-^\top \end{bmatrix}^\top \Phi \begin{bmatrix} I \\ W_-^\top \end{bmatrix} \geqslant 0 \tag{3.18}$$

where $\Phi \in \mathbb{S}^{n+T}$ is a given partitioned matrix

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \tag{3.19}$$

with $\Phi_{11} \in \mathbb{S}^n$, $\Phi_{12} \in \mathbb{R}^{n\times T}$, $\Phi_{21} = \Phi_{12}^\top$ and $\Phi_{22} \in \mathbb{S}^T$. Here we assume that $\Phi \in \mathbf{\Pi}_{n,T}$ as defined in (A.11), i.e. $\Phi_{22} \leqslant 0, \Phi \,|\, \Phi_{22} \geqslant 0$ and $\ker \Phi_{22} \subseteq \ker \Phi_{12}$.

In other words, in addition to the data $\mathcal{D}$ consisting of the measurements (U_-, X) we know that the noise on the sampling interval $[0, T]$ satisfies the inequality (3.18) for a given known partitioned matrix $\Phi \in \mathbf{\Pi}_{n,T}$.

As a result, the set of systems consistent with the data is now given by

$$\Sigma_{\mathcal{D}} = \{(A, B) \mid X_+ = AX_- + BU_- + W_- \text{ for some } W_- \text{ satisfying (3.18)}\}\,. \tag{3.20}$$

Of course, an issue is whether the set of noise matrices W_- defined by (3.18) is nonempty, equivalently, whether the set $\mathcal{Z}_T(\Phi)$ associated with the partitioned matrix Φ (as defined in (A.3)) is nonempty. This issue can be dealt with in Theorem A.5. Indeed, under the assumption $\Phi \in \mathbf{\Pi}_{n,T}$, the set $\mathcal{Z}_T(\Phi)$ is nonempty and convex. Consequently then, the set of of noise matrices W_- satisfying (3.18) is nonempty and convex.

In order to make the above quadratic inequality constraint on the matrix of noise samples more concrete, we will now look at a number of special cases.

(a) In the special case $\Phi_{12} = 0$ and $\Phi_{22} = -I$, the quadratic inequality (3.18) reduces to

$$W_- W_-^\top = \sum_{t=0}^{T-1} w(t) w(t)^\top \leqslant \Phi_{11}. \tag{3.21}$$

The inequality (3.21) can be interpreted as saying that the energy of w has a given upper bound on the time interval $[0, T-1]$.

(b) Let Ψ_{11} and Ψ_{22} be given positive definite matrices of dimensions $T \times T$ and $n \times n$, respectively, and suppose that the matrices of noise samples W_- satisfy the quadratic inequality

$$W_-^\top \Psi_{22} W_- \leqslant \Psi_{11}. \tag{3.22}$$

Note that (3.22) is a 'transposed version' of (3.21). This inequality can be reformulated as an inequality of the original form (3.18). Indeed, using two Schur complement arguments, (3.22) is equivalent to

$$\begin{bmatrix} \Psi_{11} & W_-^\top \\ W_- & \Psi_{22}^{-1} \end{bmatrix} \geqslant 0$$

which, in turn, holds if and only if $\Psi_{22}^{-1} - W_- \Psi_{11}^{-1} W_-^\top \geqslant 0$. The latter can be expressed as

$$\begin{bmatrix} I \\ W_-^\top \end{bmatrix}^\top \begin{bmatrix} \Psi_{22}^{-1} & 0 \\ 0 & -\Psi_{11}^{-1} \end{bmatrix} \begin{bmatrix} I \\ W_-^\top \end{bmatrix} \geqslant 0.$$

More generally, consider the noise model

$$\begin{bmatrix} I \\ W_- \end{bmatrix}^\top \Psi \begin{bmatrix} I \\ W_- \end{bmatrix} \geqslant 0, \tag{3.23}$$

where $\Psi \in \mathbb{S}^{n+T}$ is partitioned as

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix},$$

where $\Psi_{11} \in \mathbb{S}^n$ and $\Psi_{22} \in \mathbb{S}^T$. Assume that $\operatorname{In}(\Psi) = (T, 0, n)$. Then by Lemma A.3 the 'dual noise model' (3.23) is equivalent to the noise model (3.18) with

$$\Phi := \begin{bmatrix} 0 & -I_T \\ I_n & 0 \end{bmatrix} \Psi^{-1} \begin{bmatrix} 0 & -I_n \\ I_T & 0 \end{bmatrix},$$

in the sense that $W_- \in \mathcal{Z}_n(\Psi)$ if and only if $W_-^\top \in \mathcal{Z}_T(\Phi)$.

(c) Norm bounds on the individual noise samples $w(t)$ also give rise to bounds of the form (3.18), although this does introduce some conservatism in general. Indeed, note that for all t the pointwise norm bound $\|w(t)\|_2^2 \leqslant \varepsilon$ is equivalent to the matrix inequality $w(t)w(t)^\top \leqslant \varepsilon I$. As such, the bound (3.21) is satisfied for $\Phi_{11} = T\varepsilon I$.

(d) In some cases, we may know a priori that the noise w does not directly affect the entire state-space, but is contained in a subspace, say $\operatorname{im} E$, with E a known $n \times d$ matrix. This prior knowledge can be captured by the noise model in Assumption 3.12. Indeed, suppose that $w(t) = E\hat{w}(t)$ for all $t \in [0, T-1]$, where $\hat{w}(t) \in \mathbb{R}^d$ and $E \in \mathbb{R}^{n \times d}$ is a given matrix of full column rank. The matrix $\hat{W}_- = \hat{w}_{[0,T-1]}$ captures the noise. As before, $\hat{W}_-$ is unknown but is assumed to satisfy $\hat{W}_-^\top \in \mathcal{Z}_T(\hat{\Phi})$, where $\hat{\Phi} \in \mathbf{\Pi}_{d,T}$ is such that $\hat{\Phi}_{22} < 0$. Now, by Theorem A.7, $W_- = E\hat{W}_-$ for some $\hat{W}_-^\top \in \mathcal{Z}_T(\hat{\Phi})$ if and only if $W_-^\top \in \mathcal{Z}_T(\Phi)$, where

$$\Phi := \begin{bmatrix} E\hat{\Phi}_{11}E^\top & E\hat{\Phi}_{12} \\ \hat{\Phi}_{21}E^\top & \hat{\Phi}_{22} \end{bmatrix} \in \mathbf{\Pi}_{n,T}. \tag{3.24}$$

The conclusion is that Assumption 3.12 also covers the case in which the noise is constrained to a known subspace, which is captured by the noise bound (3.18) with Φ in (3.24).

This shows that our noise model as introduced in Assumption 3.12 encompasses many relevant special cases. This noise model will be adopted in most of the data-driven analysis and design problems based on noisy data that will be treated in the remainder of this book.

In addition to input-state systems we can also consider autonomous systems with noise. In that case we suppose that the unknown, true system is given by

$$x(t+1) = A_{\text{true}}x(t) + w(t)$$

where $x(t) \in \mathbb{R}^n$ is the state and $w(t) \in \mathbb{R}^n$ is an unknown noise term. The matrix $A_{\text{true}} \in \mathbb{R}^{n\times n}$ denotes the unknown state matrix. We embed this unknown system into the model class $\mathcal{M}$ of all autonomous systems with unknown process noise, with fixed dimension n, of the form

$$x(t+1) = Ax(t) + w(t). \tag{3.25}$$

The true system produces data $X \in \mathbb{R}^{n\times(T+1)}$ on the time interval $[0, T]$ as before, under the influence of an unknown sequence of noise samples, captured in the matrix W_- satisfying (3.18). The data $\mathcal{D}$ now consist of the state samples X and the information that the noise satisfies the bound governed by the known matrix $\Phi \in \mathbf{\Pi}_{n,T}$. In this case, the set of consistent systems $\Sigma_{\mathcal{D}}$ is given by

$$\Sigma_{\mathcal{D}} = \left\{A \in \mathbb{R}^{n\times n} \mid X_+ = AX_- + W_- \text{ for some } W_- \text{ satisfying (3.18)}\right\}.$$

3.5 Stability and stabilizability with noisy input-state data

In the present section we introduce the following four notions of informativity for stability and stabilizability of input-state systems with noise of the form (3.17).

Definition 3.13. Let (U_-, X) be data collected from (3.16) with noise matrix W_- satisfying (3.18). Let $\Sigma_{\mathcal{D}}$ be given by (3.20). Then (U_-, X) are called

(a) *informative for stability* if A is stable for all $(A, B) \in \Sigma_{\mathcal{D}}$.

(b) *informative for quadratic stability* if there exists a real matrix $P > 0$ such that $P - APA^\top > 0$ for all $(A, B) \in \Sigma_{\mathcal{D}}$.

(c) *informative for stabilizability* if every $(A, B) \in \Sigma_{\mathcal{D}}$ is stabilizable.

(d) *informative for quadratic stabilizability* if there exists a real matrix $P > 0$ such that $P - APA^\top + BB^\top > 0$ for all $(A, B) \in \Sigma_{\mathcal{D}}$.

The informativity notion in Definition 3.13 (a) expresses that all systems consistent with the data are internally stable. Property (b) requires something more, namely that all of these systems not only are stable, but also have a common Lyapunov function. Indeed, the linear matrix inequality[1] (LMI)

$$P - APA^\top > 0$$

[1] for a general reference on linear matrix inequalities, we refer to [144].

holds for $P > 0$ if and only if $P^{-1} - A^\top P^{-1} A > 0$. Thus, (b) means that all systems in $\Sigma_\mathcal{D}$ have the same (quadratic) Lyapunov function $x^\top P^{-1} x$. Property (c) entails that all systems consistent with the data are stabilizable. Lastly, property (d) is its 'quadratic' variant, which we will now explain in more detail. It turns out that for a *single* system (A, B), the existence of a $P > 0$ such that $P - APA^\top + BB^\top > 0$ is necessary and sufficient for stabilizability. This is shown in the following lemma.

Lemma 3.14. *The system (A, B) is stabilizable if and only if there exists a matrix $P > 0$ such that $P - APA^\top + BB^\top > 0$.*

Proof. To prove the 'if' statement, let $v \in \mathbb{C}^n$, $v \neq 0$, be such that $v^* A = \lambda v$ and $v^* B = 0$. Then $(1 - |\lambda|^2)\, v^* P v > 0$ which yields $|\lambda| < 1$. Using the Hautus test this implies that (A, B) is stabilizable.

Conversely, if (A, B) is stabilizable then there exists $K \in \mathbb{R}^{m \times n}$ such that $A + BK$ is stable. As a consequence there exists $P > 0$ such that

$$P - (A + BK) P (A + BK)^\top > 0.$$

By expanding this expression we obtain

$$P - APA^\top - B(KPA + \tfrac{1}{2} KPK^\top B^\top) - (A^\top PK^\top + \tfrac{1}{2} BKPK^\top) B^\top > 0.$$

It then follows immediately from the standard Finsler's lemma (see Proposition A.13) that there exists $\mu \in \mathbb{R}$ such that

$$P - APA^\top - \mu BB^\top > 0.$$

We now distinguish two cases. If $\mu \geqslant 0$, then $P - APA^\top > 0$, so obviously $P - APA^\top + BB^\top > 0$ as well. If $\mu < 0$ then

$$\tfrac{1}{-\mu} P - A(\tfrac{1}{-\mu} P) A^\top + BB^\top > 0$$

where $\frac{1}{-\mu} P > 0$. This completes the proof. □

In view of the above lemma, we see that property (d) in Definition 3.13 means that not only all systems $(A, B) \in \Sigma_\mathcal{D}$ are stabilizable, but in addition, there exists a *common* matrix $P > 0$ such that $P - APA^\top + BB^\top > 0$ holds for all $(A, B) \in \Sigma_\mathcal{D}$.

Clearly, (b) implies (a) and (d) implies (c). However, the other implications do not hold in general, as demonstrated for the case of stabilizability in the following example.

Example 3.15. Consider the case that $n = 1$, $m = 1$ and let the noise model be given by

$$\Phi = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

We consider the (unknown) true system $A_{\text{true}} = 1, B_{\text{true}} = 1$ that has generated the data $u(0) = 1$, $x(0) = 0$ and $x(1) = 1$ with the noise sample $w(0) = 0$ satisfying the noise model. In this case, the set of systems consistent with the data is $\Sigma_{\mathcal{D}} = \{(a, 1) \mid a \in \mathbb{R}\}$. Therefore, the data are informative for stabilizability. However, they are not informative for quadratic stabilizability since, for any $P > 0$, we can make the expression $P - a^2 P + 1$ negative by choosing a sufficiently large. This shows that informativity for stabilizability and informativity for quadratic stabilizability are not equivalent. ■

In the sequel we will provide conditions for the four notions of informativity defined in Definition 3.13. In particular, in Section 3.6 we will establish conditions for quadratic stability, and in Section 3.7 we will study conditions for quadratic stabilizability. The non-quadratic versions require a more sophisticated approach and their treatment is deferred to Section 3.9.

3.6 Tests for informativity for quadratic stability

In this section we will establish necessary and sufficient conditions under which the data $\mathcal{D}$ obtained from (3.16) are informative for quadratic stability.

Informativity for quadratic stability requires all systems in $\Sigma_{\mathcal{D}}$ to be *stable with a common Lyapunov function*. The conditions that we will establish will be in terms of feasibility of certain linear matrix inequalities involving the numerical data (U_-, X) and the (known) matrix Φ representing the quadratic inequality constraint on the matrix of noise samples.

Again consider the model class $\mathcal{M}$ of all noisy input-state systems with state dimension n and input dimension m of the form (3.17). Suppose we have input-state data (U_-, X) on the time interval $[0, T]$ and assume that, in addition, the possible matrices W_- of noise samples satisfy the quadratic inequality (3.18) for a given matrix $\Phi \in \mathbf{\Pi}_{n,T}$. Then the set $\Sigma_{\mathcal{D}}$ of all systems consistent with the data is given by (3.20).

We explicitly assume that the data (U_-, X) have been obtained from the unknown system (3.16). In other words, we assume that $(A_{\text{true}}, B_{\text{true}}) \in \Sigma_{\mathcal{D}}$. In particular this implies that the set $\Sigma_{\mathcal{D}}$ is nonempty.

We will now outline our strategy for characterizing informativity for quadratic stability defined in Definition 3.13 (b). Let $(A, B) \in \Sigma_{\mathcal{D}}$ and rewrite the equation defining (3.20) as

$$W_- = X_+ - AX_- - BU_-. \tag{3.26}$$

Recall that by Assumption 3.12, we have

$$\begin{bmatrix} I \\ W_-^\top \end{bmatrix}^\top \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I \\ W_-^\top \end{bmatrix} \geqslant 0.$$

By substitution of (3.26), this yields

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} \geqslant 0. \tag{3.27}$$

This shows that A and B satisfy a *quadratic matrix inequality* (QMI) of the form (3.27). In fact, the set $\Sigma_{\mathcal{D}}$ of all systems consistent with the data can be equivalently characterized in terms of (3.27), as asserted in the following lemma.

Lemma 3.16. *We have that* $\Sigma_{\mathcal{D}} = \{(A, B) \mid (3.27)$ *is satisfied*$\}$.

Proof. Suppose that $(A, B) \in \Sigma_{\mathcal{D}}$. Then (3.26) is satisfied for some W_- satisfying (3.18). This means that (3.27) holds. Therefore,

$$\Sigma_{\mathcal{D}} \subseteq \{(A, B) \mid (3.27) \text{ is satisfied}\}.$$

To prove the reverse inclusion, let (A, B) be such that (3.27) is satisfied. Define $W_- := X_+ - AX_- - BU_-$. By (3.27), W_- satisfies Assumption (3.12). Since the equation $X_+ = AX_- + BU_- + W_-$ holds for (A, B) by construction, we conclude that $(A, B) \in \Sigma_{\mathcal{D}}$. ☐

By Lemma 3.16 the set $\Sigma_{\mathcal{D}}$ of systems consistent with the data is characterized by a quadratic matrix inequality in (A, B). Next, suppose that we fix a Lyapunov matrix $P > 0$. Note that the inequality $P - APA^\top > 0$ is equivalent to

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} > 0 \tag{3.28}$$

which is yet another quadratic matrix inequality in A and B. Therefore, characterizing informativity for quadratic stability essentially boils down to understanding under which conditions there exists a matrix $P > 0$ such that the quadratic matrix inequality (3.28) holds for all (A, B) satisfying the quadratic matrix inequality (3.27). This naturally leads to the following fundamental question.

When does one QMI imply another QMI?

A detailed discussion on this question can be found in the appendix (see Section A.3). We will now apply the theory developed there in order to obtain necessary and sufficient conditions on the data to be informative for quadratic stability. To this end, for given $P = P^\top > 0$ define the partitioned matrices

$$M = \left[\begin{array}{c:c} M_{11} & M_{12} \\ \hdashline M_{12}^\top & M_{22} \end{array}\right] := \left[\begin{array}{c:cc} P & 0 & 0 \\ \hdashline 0 & -P & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{3.29}$$

$$N = \left[\begin{array}{c:c} N_{11} & N_{12} \\ \hdashline N_{12}^\top & N_{22} \end{array}\right] := \left[\begin{array}{cc} I & X_+ \\ \hdashline 0 & -X_- \\ 0 & -U_- \end{array}\right] \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \left[\begin{array}{cc} I & X_+ \\ \hdashline 0 & -X_- \\ 0 & -U_- \end{array}\right]^\top . \tag{3.30}$$

Recall that informativity for quadratic stability entails deciding whether there exists P so that (3.28) holds for all (A, B) satisfying (3.27). In terms of the matrices M and N as defined above, we thus have to decide whether

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} > 0 \quad \text{for all } Z \in \mathbb{R}^{(n+m)\times n} \text{ such that } \begin{bmatrix} I \\ Z \end{bmatrix}^\top N \begin{bmatrix} I \\ Z \end{bmatrix} \geqslant 0, \tag{3.31}$$

where Z is given by

$$Z := \begin{bmatrix} A^\top \\ B^\top \end{bmatrix}.$$

Using the sets defined in (A.3) and (A.12) (see Section A.2), condition (3.31) can be equivalently restated as

$$\mathcal{Z}_{n+m}(N) \subseteq \mathcal{Z}^+_{n+m}(M). \tag{3.32}$$

Strict matrix S-lemmas (Theorem A.20 and Corollary A.23) give conditions such that (3.32) holds. In fact, these theorems give *necessary and sufficient* conditions for (3.32) to hold. Obtaining sufficient conditions is straightforward. However, showing that these conditions are also necessary requires the full force of these results, for which it is required to verify their assumptions on N and M as given by (3.29) and (3.30).

We will start off with the assumptions of Corollary A.23, which requires that $N_{22} \leqslant 0$, $\ker N_{22} \subseteq \ker N_{12}$, $N \,|\, N_{22} \geqslant 0$ and $M_{22} \leqslant 0$. Obviously, $M_{22} \leqslant 0$ since $P > 0$. Also,

$$N_{22} = \begin{bmatrix} X_- \\ U_- \end{bmatrix} \Phi_{22} \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top \leqslant 0$$

because $\Phi_{22} \leqslant 0$ by the assumption that $\Phi \in \mathbf{\Pi}_{n,T}$. We also see that

$$\begin{aligned} \ker N_{22} &= \ker \left(\Phi_{22} \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top \right) \\ \ker N_{12} &= \ker \left((\Phi_{12} + X_+ \Phi_{22}) \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top \right). \end{aligned}$$

Using the assumption $\ker \Phi_{22} \subseteq \ker \Phi_{12}$ this implies $\ker N_{22} \subseteq \ker N_{12}$. Next we prove that $N \mid N_{22} \geqslant 0$. Indeed, recall that $\mathcal{Z}_{n+m}(N) = \Sigma_{\mathcal{D}}$ is nonempty because of the assumption that it contains the true system $(A_{\text{true}}, B_{\text{true}})$. Since in addition $N_{22} \leqslant 0$, by applying inequality (A.10) with $\Pi = N$ we get $N \mid N_{22} \geqslant 0$.

Corollary A.23 now asserts that (3.32) holds if and only if there exist scalars $\alpha \geqslant 0$ and $\beta > 0$ such

$$M - \alpha N \geqslant \begin{bmatrix} \beta I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.33}$$

Of course, the matrix P that appears in M is not given. However, by the above discussion, the data (U_-, X) are informative for quadratic stability *if and only if* there exist an $n \times n$ matrix $P > 0$, and two scalars $\alpha \geqslant 0$ and $\beta > 0$ such that (3.33) holds. Note that (3.33) is a linear matrix inequality in P, α and β. Due to the particular structure of M and N, it turns out that the scalar α must in fact be positive, and therefore the inequality (3.33) can be scaled by $\frac{1}{\alpha}$. Thus we obtain the following theorem that characterizes informativity for quadratic stability in terms of feasibility of a linear matrix inequality composed of the data (U_-, X) and the given Φ-matrix.

Theorem 3.17. *The data (U_-, X) are informative for quadratic stability if and only if there exists an $n \times n$ matrix $P > 0$, and a scalar $\beta > 0$ satisfying*

$$\begin{bmatrix} P - \beta I & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top \geqslant 0. \tag{3.34}$$

Proof. By Corollary A.23, the inclusion (3.32) holds if and only if there exist scalars $\alpha \geqslant 0$ and $\beta > 0$ such that

$$\begin{bmatrix} P - \beta I & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & 0 \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top \geqslant 0.$$

Taking a look at the $(2,2)$ block in this inequality, we see that we must have $-P - \alpha X_- \Phi_{22} X_-^\top \geqslant 0$. Since $P > 0$ this yields $\alpha > 0$. By scaling P and β by $\frac{1}{\alpha}$ we arrive at the inequality (3.34). ☐

Next, we will show that under the additional assumption that

$$\operatorname{rank} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = n + m \tag{3.35}$$

we can apply Theorem A.20 to obtain alternative conditions for informativity for quadratic stability. Indeed, in Theorem A.20 the assumptions are that $N_{22} < 0$

and $N \,|\, N_{22} \geqslant 0$. As we have already proven above, the second condition is indeed satisfied. The condition $N_{22} < 0$ holds if we impose the condition $\Phi_{22} < 0$ and the full row rank condition (3.35).

In that case, Theorem A.20 asserts that (3.32) holds if and only if there exist a scalar $\alpha \geqslant 0$ such that $M - \alpha N > 0$. As in the previous, the scalar α is necessarily positive, and therefore this inquality can be scaled. This leads to the following theorem.

Theorem 3.18. *Assume that $\Phi_{22} < 0$ and the full rank condition (3.35) holds. Then the data (U_-, X) are informative for quadratic stability if and only if there exists an $n \times n$ matrix $P > 0$ satisfying*

$$\begin{bmatrix} P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top > 0. \tag{3.36}$$

In contrast to the notion of informativity for stability in the exact data setting, as studied in Theorem 3.7, we note that informativity for quadratic stability does not imply that A_{true} is uniquely identifiable from the data. However, by inspection of the $(2,2)$-block of (3.36), we do note that a necessary condition for informativity for quadratic stability is that X_- has full row rank.

3.7 A test for informativity for quadratic stabilizability

In this section we will derive tests for verifying informativity for quadratic stabilizability, as defined in Definition 3.13 (d). Again, these tests will involve feasibility of linear matrix inequalities composed of the data (U_-, X) and the given matrix Φ.

As in Section 3.6, we consider the model class $\mathcal{M}$ of all noisy input-state systems (3.17) with state dimension n and input dimension m. Also, we assume that we have input-state data (U_-, X) on the time interval $[0, T]$, with the prior information that the possible matrices W_- of noise samples satisfy the quadratic inequality (3.18) for a given $\Phi \in \mathbf{\Pi}_{n,T}$. Then the set $\Sigma_\mathcal{D}$ of all systems consistent with the data is given by (3.20). Recall that we assume that the unknown system $(A_{\text{true}}, B_{\text{true}})$ is in $\Sigma_\mathcal{D}$, which is therefore nonempty.

Recall from Lemma 3.16 that $\Sigma_\mathcal{D}$ is equal to the set of all (A, B) that satisfy the quadratic matrix inequality (3.27). On the other hand, also the inequality $P - APA^\top + BB^\top > 0$ can be reformulated as a quadratic matrix inequality in A and B. Indeed, for given $P > 0$, this matrix inequality holds if and only if

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} > 0 \tag{3.37}$$

Hence, characterizing informativity for quadratic stabilizability amounts to finding necessary and sufficient conditions under which the quadratic matrix inequality (3.37) holds for all (A, B) satisfying the quadratic matrix inequality (3.27). As before, let N be given by (3.30). For given $P = P^\top > 0$ define

$$M = \left[\begin{array}{c:c} M_{11} & M_{12} \\ \hdashline M_{12}^\top & M_{22} \end{array}\right] := \left[\begin{array}{c:cc} P & 0 & 0 \\ \hdashline 0 & -P & 0 \\ 0 & 0 & I \end{array}\right].$$

Then, again, we have to find conditions in terms of the matrices M and N such that $\mathcal{Z}_{n+m}(N) \subseteq \mathcal{Z}_{n+m}^+(M)$. Note that in this case not all assumptions of Corollary A.23 can be satisfied. In particular the condition $M_{22} \leqslant 0$ never holds. However we are able to apply Theorem A.20 under the additional assumption $\Phi_{22} < 0$ on the noise model and the full rank condition (3.35) on the data. Indeed, under these assumptions we have $N_{22} < 0$, so Theorem A.20 states that $\mathcal{Z}_{n+m}(N) \subseteq \mathcal{Z}_{n+m}^+(M)$ if and only if there exists a scalar $\alpha \geqslant 0$ such that $M - \alpha N > 0$. Thus we obtain the following characterization of informativity for quadratic stabilizability.

Theorem 3.19. *Assume that $\Phi_{22} < 0$ and the full rank condition (3.35) holds. Then the data (U_-, X) are informative for quadratic stabilizability if and only if there exist a scalar $\alpha \geqslant 0$ and an $n \times n$ matrix $P > 0$ satisfying*

$$\begin{bmatrix} P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & I \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top > 0. \tag{3.38}$$

Finally, we note that (3.38) can be further simplified. In particular, (3.38) is equivalent to a linear matrix inequality of size $2n \times 2n$. This is made precise in the following corollary.

Corollary 3.20. *Assume that $\Phi_{22} < 0$ and the full rank condition (3.35) holds. Then the data (U_-, X) are informative for quadratic stabilizability if and only if there exists a real matrix $P > 0$ such that*

$$\begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix}^\top > 0. \tag{3.39}$$

Proof. By Theorem 3.19, the data (U_-, X) are informative for quadratic stabilizability if and only if there exists a matrix $P > 0$ and a real number $\alpha \geqslant 0$ such that (3.38) holds.

By zooming in on the $(2, 2)$-block of that inequality, we observe that the inequality can hold only if $\alpha > 0$. Hence, feasibility of (3.38) is equivalent to the existence of $P > 0$ and $\mu > 0$ such that

$$\begin{bmatrix} P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & \mu I \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top > 0. \tag{3.40}$$

Note that the upper left 2×2 block of the matrix in (3.40) equals the matrix in (3.39). Therefore, it is clear that informativity for quadratic stabilizability implies (3.39). Conversely, if (3.39) is feasible then there exists a sufficiently large $\mu > 0$ such that (3.40) holds, i.e., (U_-, X) are informative for quadratic stabilizability. This proves the corollary. □

An interesting consequence of Corollary 3.20 is that the condition for informativity for quadratic stabilizability in (3.39) is precisely the same as the condition for informativity for stability of the state data X (see (3.44)). We do note, however, that the state data in the current section have been obtained from a different system (3.16) including a control input u.

3.8 Stability of autonomous systems using noisy state data

At this point, the reader may wonder whether the stability of *autonomous* systems (analogous to the last part of Section 3.3) can also be studied in the noisy data setting. This indeed turns out to be the case. As explained in Section 3.4, for this we consider the system

$$x(t+1) = A_{\text{true}} x(t) + w(t). \tag{3.41}$$

This system produces the data $X \in \mathbb{R}^{n \times (T+1)}$ as before, under the influence of an unknown sequence of noise samples, captured in the matrix W_- satisfying (3.18). The data $\mathcal{D}$ now consists of the state samples X and the information that the noise satisfies the bound governed by the known matrix $\Phi \in \mathbf{\Pi}_{n,T}$. In this case, the set of consistent systems is given by

$$\Sigma_{\mathcal{D}} = \left\{ A \in \mathbb{R}^{n \times n} \mid X_+ = AX_- + W_- \text{ for some } W_- \text{ satisfying (3.18)} \right\}.$$

Analogous to Definition 3.13 we now have the following definition of informativity for stability and quadratic stability of the state data X.

Definition 3.21. The state data X, obtained from (3.41) with noise samples W_- satisfying (3.18), are called

(a) *informative for stability* if all matrices $A \in \Sigma_{\mathcal{D}}$ are stable.

(b) *informative for quadratic stability* if there exists a matrix $P > 0$ such that $P - APA^\top > 0$ for all matrices $A \in \Sigma_{\mathcal{D}}$.

In the present section we will give necessary and sufficient conditions for informativity for quadratic stability for autonomous systems as defined in Definition 3.21. A discussion on informativity for stability without the requirement of a common Lyapunov fuction will be provided in Section 3.9.

To treat informativity for quadratic stability, we will need the following two lemmas.

Lemma 3.22. *Let the data X be obtained from (3.41) where the noise samples satisfy (3.18) for a given $\Phi \in \mathbf{\Pi}_{n,T}$. Then*

$$\begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix}^\top \in \mathbf{\Pi}_{n,n}. \tag{3.42}$$

Proof. Let N be the matrix in (3.42) and note that N equals

$$\begin{bmatrix} \Phi_{11} + X_+\Phi_{21} + \Phi_{12}X_+^\top + X_+\Phi_{22}X_+^\top & -\Theta X_-^\top \\ -X_-\Theta^\top & X_-\Phi_{22}X_-^\top \end{bmatrix}$$

where

$$\Theta := \Phi_{12} + X_+\Phi_{22}. \tag{3.43}$$

Since $\Phi_{22} \leqslant 0$, we have that $X_-\Phi_{22}X_-^\top \leqslant 0$. In addition, $\ker(X_-\Phi_{22}X_-^\top) = \ker(\Phi_{22}X_-^\top)$. Since $\ker \Phi_{22} \subseteq \ker \Phi_{12}$, it holds that $\Phi_{12} = M\Phi_{22}$ for some real matrix M. We conclude that $\ker(X_-\Phi_{22}X_-^\top) \subseteq \ker(-\Theta X_-^\top)$. Since $A_{\text{true}}^\top \in \mathcal{Z}_n(N)$, it follows that $N \,|\, (X_-\Phi_{22}X_-^\top) \geqslant 0$ by (A.10). This proves the lemma. ☐

Lemma 3.23. *The data X are informative for quadratic stability only if X_- has full row rank.*

Proof. Assume that the data X are informative for quadratic stability. In particular this implies that all systems in $\Sigma_\mathcal{D}$ are stable. Let $\xi \in \mathbb{R}^n$ be such that $\xi^\top X_- = 0$. Then $A + \alpha\xi\xi^\top \in \Sigma_\mathcal{D}$ for any $A \in \Sigma_\mathcal{D}$ and $\alpha \in \mathbb{R}$. Hence $A + \alpha\xi\xi^\top$ is stable for all $\alpha \in \mathbb{R}$. This implies that for all $\alpha \in \mathbb{R}$, the trace of $A + \alpha\xi\xi^\top$ satisfies $\operatorname{tr}(A + \alpha\xi\xi^\top) < n$. In other words, for all $\alpha \in \mathbb{R}$, $\operatorname{tr}(A) + \alpha\|\xi\|^2 < n$. This implies that $\xi = 0$, thus X_- has full row rank. ☐

Based on the previous two lemmas, we can state the following theorem that provides a necessary and sufficient condition under which the data X are informative for quadratic stability.

Theorem 3.24. *Let the data X be obtained from (3.41) where the noise satisfies (3.18) with $\Phi \in \mathbf{\Pi}_{n,T}$ and $\Phi_{22} < 0$. Then X are informative for quadratic stability if and only if there exists a real matrix $P > 0$ such that*

$$\begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix}^\top > 0. \tag{3.44}$$

Proof. The 'if' part follows directly by taking $A \in \Sigma_{\mathcal{D}}$ and multiplying (3.44) from the left by $\begin{bmatrix} I & A \end{bmatrix}$ and from the right by its transposed.

Next, we focus on proving the 'only if' part. Thus, suppose that there exists a $P > 0$ such that $P - APA^\top > 0$ for all $A \in \Sigma_{\mathcal{D}}$. The matrix X_- has full row rank by Lemma 3.23. This implies that $X_-\Phi_{22}X_-^\top < 0$. Moreover, by Lemma 3.22, (3.42) holds. Therefore, by Theorem A.20, there exists a real number $\alpha \geqslant 0$ such that

$$\begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix}^\top > 0.$$

In particular, $-P - \alpha X_-\Phi_{22}X_-^\top > 0$, so $\alpha > 0$. Therefore, by rescaling P by $\frac{1}{\alpha}$, we conclude that the linear matrix inequality (3.44) is feasible. □

3.9 Informativity beyond common Lyapunov functions

In this section we will provide conditions for informativity for stability and stabilizability for noisy input-state systems and for noisy autonomous systems as defined in Definition 3.13 (a), Definition 3.13 (c) and Definition 3.21 (a). In all these cases it turns out that, under certain technical assumptions, informativity for stability and quadratic stability are equivalent, and allow a test in terms of the given data.

3.9.1 Stability from noisy state data

We start off with the case of state data X obtained from (3.41), and formulate a theorem that characterizes informativity for stability of autonomous systems as defined in Definition 3.21 (a). Also, a numerical example will be given to illustrate this result.

It turns out that under a certain eigenvalue condition on a matrix obtained from the data, informativity for stability and quadratic stability are *equivalent*. Moreover, the theorem provides an alternative for Theorem 3.24 by giving a condition for informativity for quadratic stability that does not rely on the solvability of a linear matrix inequality. In fact, condition (c) of the following theorem involves checking that X_- has full row rank, and that two other matrices are negative definite and stable, respectively. In the following, let N be the matrix in (3.42) and let Θ be as defined in (3.43).

Theorem 3.25. *Let the data X be obtained from (3.41) where the noise satisfies (3.18) with $\Phi \in \mathbf{\Pi}_{n,T}$ and $\Phi_{22} < 0$. For $\lambda \in \mathbb{C}$ define*

$$\Psi(\lambda) := \begin{bmatrix} I \\ \lambda I \end{bmatrix}^* N \begin{bmatrix} I \\ \lambda I \end{bmatrix}. \tag{3.45}$$

Assume that $\Psi(1)$ *is invertible and the matrix*

$$\begin{bmatrix} 0 & \Psi(1)^{-1} \\ \Psi(-1) & 2(\Theta X_-^\top - X_-\Theta^\top)\Psi(1)^{-1} \end{bmatrix} \tag{3.46}$$

has no eigenvalues on the imaginary axis. Then the following statements are equivalent:

(a) *The data* X *are informative for quadratic stability.*

(b) *The data* X *are informative for stability.*

(c) X_- *has full row rank,* $\Psi(1) < 0$*, and the matrix* $(X_-\Phi_{22}X_-^\top)^{-1}X_-\Theta^\top$ *is stable.*

Before we are able to prove this theorem, we need a lemma, a proposition and a corollary, as discussed next. In what follows let $\mathcal{C}$ denote the unit circle in the complex plane $\mathbb{C}$.

Lemma 3.26. *Let* $Q \in \mathbb{S}^n$ *and* $R \in \mathbb{R}^{n\times n}$*. Define* $\Psi(\lambda) := Q + \lambda R + \lambda^{-1}R^\top$*. We have that* $\Psi(\lambda) < 0$ *for all* $\lambda \in \mathcal{C}$ *if and only if* $\Psi(1) = Q + R + R^\top < 0$ *and the matrix*

$$\begin{bmatrix} 0 & \Psi(1)^{-1} \\ \Psi(-1) & -2(R - R^\top)\Psi(1)^{-1} \end{bmatrix} \tag{3.47}$$

has no eigenvalues on the imaginary axis.

Proof. First note that

$$\begin{bmatrix} 0 & \Psi(1)^{-1} \\ \Psi(-1) & -2(R - R^\top)\Psi(1)^{-1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mu \begin{bmatrix} x \\ y \end{bmatrix}$$

for $\mu \in \mathbb{C}$ and $x, y \in \mathbb{C}^n$ if and only if $y = \mu\Psi(1)x$ and $(\mu^2\Psi(1) + 2\mu(R - R^\top) - \Psi(-1))x = 0$. We substitute $\Psi(1) = Q + R + R^\top$ and $\Psi(-1) = Q - R - R^\top$ and rearrange terms to show that the latter equation is equivalent to

$$\left(Q + \frac{\mu+1}{\mu-1}R + \frac{\mu-1}{\mu+1}R^\top\right)x = 0.$$

Finally, by substituting $\lambda := \frac{\mu+1}{\mu-1}$, the latter is equivalent to $x \in \ker \Psi(\lambda)$. Note that $M : i\mathbb{R} \to \mathcal{C} \setminus \{1\}$ defined by $M : \mu \mapsto \frac{\mu+1}{\mu-1}$ is a bijection between the imaginary axis and $\mathcal{C} \setminus \{1\}$. As such, the above discussion shows that (3.47) has no eigenvalues on the imaginary axis if and only if $\Psi(\lambda)$ is nonsingular for all $\lambda \in \mathcal{C} \setminus \{1\}$.

Next, we prove the 'if' part. Thus, suppose that $\Psi(1) < 0$ and (3.47) has no imaginary axis eigenvalues. This implies that $\Psi(\lambda)$ is nonsingular for all $\lambda \in \mathcal{C}$.

We note that $\Psi(\lambda)$ is Hermitian and thus has only real eigenvalues for all $\lambda \in \mathcal{C}$. Moreover, $\Psi(\lambda)$ is a continuous function of λ. It thus follows that the largest eigenvalue of $\Psi(\lambda)$ is a continuous function of λ.

Now, suppose that there exists a $\lambda \in \mathcal{C}$ such that $\Psi(\lambda) \not< 0$. By continuity of the largest eigenvalue of $\Psi(\lambda)$ and the fact that $\Psi(1) < 0$, there exists a particular value $\bar{\lambda} \in \mathcal{C}$ such that $\Psi(\bar{\lambda})$ is singular. This is a contradiction, and we thus conclude that $\Psi(\lambda) < 0$ for all $\lambda \in \mathcal{C}$.

Next, to prove the 'only if' part, suppose that $\Psi(\lambda) < 0$ for all $\lambda \in \mathcal{C}$. Clearly, this implies that $\Psi(1) < 0$. Also, $\Psi(\lambda)$ is nonsingular for all $\lambda \in \mathcal{C}$, thus, in particular, for all $\lambda \in \mathcal{C} \setminus \{1\}$. We conclude that (3.47) has no imaginary eigenvalues. This proves the lemma. □

The following proposition is a discrete-time version of the well-known Kalman-Yakubovich-Popov (KYP) lemma. It can be obtained as a special case from the generalized KYP lemma [80] (see also [127]).

Proposition 3.27. *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $N \in \mathbb{S}^{n+m}$. There exists a matrix $P \in \mathbb{S}^n$ such that*

$$\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} - N > 0$$

if and only if for all $\lambda \in \mathcal{C}$ the following implication holds:

$$\begin{bmatrix} A - \lambda I & B \end{bmatrix} v = 0 \text{ for } v \in \mathbb{C}^{n+m} \setminus \{0\} \implies v^* N v < 0.$$

Corollary 3.28. *Let $N \in \mathbb{S}^{2n}$ be partitioned as*

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

where N_{11}, N_{12}, N_{21} and N_{22} are $n \times n$ matrices. For $\lambda \in \mathcal{C}$ define the mapping

$$\Psi(\lambda) := N_{11} + N_{22} + \lambda N_{12} + \lambda^{-1} N_{21}.$$

Then there exists a $P \in \mathbb{S}^n$ such that

$$\begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} - N > 0 \tag{3.48}$$

if and only if $\Psi(1) < 0$ and the matrix

$$\begin{bmatrix} 0 & \Psi(1)^{-1} \\ \Psi(-1) & -2(N_{12} - N_{21})\Psi(1)^{-1} \end{bmatrix} \tag{3.49}$$

has no eigenvalues on the imaginary axis.

Proof. The result follows by consecutively applying Proposition 3.27 with $m = n$, $A = 0$ and $B = I$, and Lemma 3.26 with $Q = N_{11} + N_{22}$ and $R = N_{12}$. □

Finally, we are in the position to prove Theorem 3.25.

Proof of Theorem 3.25. The implication (a) $\implies$ (b) is clear. Next, we prove that (b) $\implies$ (c). First, we see that (b) implies full row rank of X_- by Lemma 3.23, so that $X_- \Phi_{22} X_-^\top < 0$. Let N be the matrix in (3.42) and note that $N \in \mathbf{\Pi}_{n,n}$ by Lemma 3.22. Therefore, $N \,|\, N_{22} \geqslant 0$ and thus $-N_{22}^{-1} N_{21} \in \mathcal{Z}_n(N)$. Since $-N_{12} N_{22}^{-1} \in \Sigma_{\mathcal{D}}$, it is stable. We thus conclude that

$$-N_{22}^{-1} N_{21} = (X_- \Phi_{22} X_-^\top)^{-1} X_- \Theta^\top$$

is also stable. Finally, we will prove that $\Psi(1) < 0$. Suppose on the contrary that there exists a nonzero vector $x \in \mathbb{R}^n$ such that $x^\top \Psi(1) x \geqslant 0$, equivalently,

$$\begin{bmatrix} x \\ x \end{bmatrix}^\top N \begin{bmatrix} x \\ x \end{bmatrix} \geqslant 0. \tag{3.50}$$

By Lemma A.15 there exists a matrix $Z \in \mathcal{Z}_n(N)$ such that

$$\begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} I \\ Z \end{bmatrix} x. \tag{3.51}$$

Since $Zx = x$, Z has an eigenvalue 1 and is thus not stable. However, this is a contradiction because $Z^\top \in \Sigma_{\mathcal{D}}$. As such, we conclude that $\Psi(1) < 0$.

Finally, we prove that (c) $\implies$ (a). Since $\Psi(1) < 0$ and (3.46) has no imaginary eigenvalues, we conclude by Corollary 3.28 that there exists a $P \in \mathbb{S}^n$ satisfying (3.48) with N given in (3.42). We will now show that $P > 0$. To this end, note that $Z := (X_- \Phi_{22} X_-^\top)^{-1} X_- \Theta^\top \in \mathcal{Z}_n(N)$. Hence, by multiplying (3.48) from the left by $\begin{bmatrix} I & Z^\top \end{bmatrix}$ and from the right by its transposed we obtain $P - Z^\top P Z > 0$, that is, P satisfies the Lyapunov equation $P - Z^\top P Z = Y$ for some matrix $Y > 0$. Since Z is stable by hypothesis, the solution P is unique and given by $P = \sum_{k=0}^{\infty} (Z^\top)^k Y Z^k$. We observe that $P > 0$ because $Y > 0$. Therefore, we conclude that there exists a $P > 0$ satisfying (3.48) with N given by (3.42). According to Theorem 3.24, the data X are therefore informative for quadratic stability. □

Example 3.29. Consider the unweighted Laplacian matrix $L \in \mathbb{S}^n$ associated with an undirected cycle graph. This means that all diagonal entries of L are equal to 2, while the off-diagonal entries are $L_{ij} = -1$ if $|i - j| = 1$ or $|i - j| = n - 1$, and $L_{ij} = 0$ for all other $i \neq j$. We study the discrete-time consensus protocol

$$x(t+1) = (I - aL)x(t)$$

where $a \in (0, \frac{1}{2})$. This protocol is studied in the presence of a so-called *stubborn agent*, assumed to be node n, who keeps its state at the contant value $x_n(t) = 0$ for all t. The resulting dynamics are thus

$$x(t+1) = \begin{bmatrix} I - aL_g & 0_{(n-1),1} \\ 0_{1,(n-1)} & 0 \end{bmatrix} x(t) \tag{3.52}$$

where $L_g \in \mathbb{R}^{(n-1)\times(n-1)}$ is the *grounded Laplacian*, i.e., the upper left $(n-1) \times (n-1)$ submatrix of L. It is well-known that (3.52) is stable (i.e., all agents reach consensus and converge to the zero state of the stubborn agent) if and only if the network graph underlying L is *connected*.

With complete knowledge of the (cycle) topology of the network, we are thus immediately able to conclude stability of (3.52). In this example, we aim to verify stability without knowledge of the graph, but using data instead. To this end, we assume that state data are obtained from (3.41) with A_{true} equal to the matrix in (3.52). The noise only directly affects node 1, that is, we assume that $|w_1(t)| \leqslant \varepsilon$ and $w_i(t) = 0$ for all $i = 2, \dots, n$. This implies that the noise satisfies the noise model (3.18) with

$$\Phi = \begin{bmatrix} \varepsilon^2 TEE^\top & 0 \\ 0 & -I \end{bmatrix},$$

where $E \in \mathbb{R}^n$ denotes the first standard basis vector of $\mathbb{R}^n$.

We take $n = 500$. In this case, quadratic stability is challenging to verify by solving (3.44) using LMI solvers, since a large number of 125250 decision variables is involved. Therefore, we apply Theorem 3.25. We collect 30 data sets according to $X_+^i = A_{\text{true}} X_-^i + W_-^i$, where $i = 1, 2, \dots, 30$. The matrices X_-^i, X_+^i and W_-^i are in $\mathbb{R}^{n\times 100}$. Each entry of the first row of W_-^i is selected uniformly at random from $\{-\varepsilon, \varepsilon\}$, while all other entries are zero. The initial state of each experiment, i.e., the first column of X_-^i, is drawn randomly from a standard Gaussian distribution, and is scaled by n. The combined data matrices of all 30 experiments are given by $X_- = \begin{bmatrix} X_-^1 & \cdots & X_-^{30} \end{bmatrix}$ and $X_+ = \begin{bmatrix} X_+^1 & \cdots & X_+^{30} \end{bmatrix}$. Next, we verify the conditions of Theorem 3.25 for various levels of ε. For each ε we generate 100 data matrices X_- and X_+. For the assumption of Theorem 3.25 and each of the three conditions of Theorem 3.25(c), we record the number of data sets for which the condition was satisfied. The results are displayed in Table 3.1. Here 'Eigs. (3.46)' refers to the conditions that $\Psi(1)$ is invertible and (3.46) does not have imaginary eigenvalues. Moreover, 'stable' refers to the matrix $(X_-\Phi_{22}X_-^\top)^{-1}X_-\Theta^\top$ being stable.

We see that the assumption on the eigenvalues of (3.46) is satisfied for every data set and all levels of ε. Also, the rank of X_- is always 500 and the matrix $(X_-\Phi_{22}X_-^\top)^{-1}X_-\Theta^\top$ is stable in all experiments. The condition $\Psi(1) < 0$, however, is not satisfied in all experiments. We see that for a small value of

ε	Eigs. (3.46)	$\Psi(1) < 0$	rank $X_- = n$	stable
0.10	100%	100%	100%	100%
0.15	100%	95%	100%	100%
0.20	100%	75%	100%	100%
0.25	100%	55%	100%	100%
0.30	100%	33%	100%	100%

Table 3.1: Percentage of trials in which the different conditions of Theorem 3.25 hold, for various levels of ε.

$\varepsilon = 0.10$, $\Psi(1) < 0$ for all 100 data sets. This implies that each of these data sets are informative for stability, by Theorem 3.25. The percentage of data sets for which $\Psi(1) < 0$ decreases as ε increases. For $\varepsilon = 0.30$ only 33 of the 100 data sets were informative for stability.

It is of interest to observe that even in the case of the larger noise bound $\varepsilon = 0.30$, the matrix $(X_-\Phi_{22}X_-^\top)^{-1}X_-\Theta^\top$ is always stable. This highlights the 'price of robustness': even though there *exists* a system in $\Sigma_\mathcal{D}$ that is stable, we cannot always conclude that *all* systems in $\Sigma_\mathcal{D}$ are stable. ■

3.9.2 Stability from noisy input-state data

Next, we consider data (U_-, X) obtained from the noisy input-state system (3.16) and study informativity for stability as defined in Definition 3.13 (a). First note that Lemma 3.22 can be extended straightforwardly to the input-state case, which we present here without proof.

Lemma 3.30. *Let the data (U_-, X) be obtained from (3.16) where the noise samples satisfy (3.18) with $\Phi \in \mathbf{\Pi}_{n,T}$. Then*

$$\begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top \in \mathbf{\Pi}_{n,n+m}. \tag{3.53}$$

Again, let the matrix Θ be as defined in (3.43) and let N be the matrix in (3.53).

Theorem 3.31. *Let the data (U_-, X) be obtained from (3.16) where the noise satisfies (3.18) with $\Phi \in \mathbf{\Pi}_{n,T}$ and $\Phi_{22} < 0$. Assume that the data (U_-, X)*

satisfy the full rank condition (3.35). For $\lambda \in \mathbb{C}$ define the mapping Ψ by

$$\Psi(\lambda) := \begin{bmatrix} I & 0 \\ \lambda I & 0 \\ 0 & I \end{bmatrix}^* N \begin{bmatrix} I & 0 \\ \lambda I & 0 \\ 0 & I \end{bmatrix}. \tag{3.54}$$

Also define

$$R := \begin{bmatrix} -\Theta X_-^\top & 0 \\ U_-\Phi_{22}X_-^\top & 0 \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)} \tag{3.55}$$

Suppose that $\Psi(1)$ is invertible and the matrix

$$\begin{bmatrix} 0 & \Psi(1)^{-1} \\ \Psi(-1) & -2(R-R^\top)\Psi(1)^{-1} \end{bmatrix} \tag{3.56}$$

has no eigenvalues on the imaginary axis. Then the following statements are equivalent:

(a) *(U_-, X) are informative for quadratic stability*

(b) *(U_-, X) are informative for stability*

(c) *$\Psi(1) < 0$ and the matrix*

$$\begin{bmatrix} I_n & 0 \end{bmatrix} \left(\begin{bmatrix} X_- \\ U_- \end{bmatrix} \Phi_{22} \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top \right)^{-1} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \Theta^\top \tag{3.57}$$

is stable.

Proof. The implication (a) $\implies$ (b) is clear. Next, we prove that (b) $\implies$ (c). By Lemma 3.30, $N \in \mathbf{\Pi}_{n,n+m}$ and therefore $N \mid N_{22} \geqslant 0$. By assumption, $\Phi_{22} < 0$ and therefore

$$N_{22} = \begin{bmatrix} X_- \\ U_- \end{bmatrix} \Phi_{22} \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top < 0.$$

Thus, N_{22} is invertible and $N_{22}^{-1}N_{21} \in \mathcal{Z}_{n+m}(N)$. Therefore, $(A, B) \in \Sigma_{\mathcal{D}}$, where

$$\begin{bmatrix} A^\top \\ B^\top \end{bmatrix} := N_{22}^{-1}N_{21}.$$

Since the data (U_-, X) are informative for stability, A is stable and hence $A^\top$ is stable. Stability of the matrix (3.57) then follows from the fact that it is equal to $\begin{bmatrix} I_n & 0 \end{bmatrix} N_{22}^{-1}N_{21} = A^\top$. Next we will prove that $\Psi(1) < 0$. Suppose, on the contrary, that there exist vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, not both zero, such that

$$\begin{bmatrix} x \\ y \end{bmatrix}^\top \Psi(1) \begin{bmatrix} x \\ y \end{bmatrix} \geqslant 0.$$

That is,

$$\begin{bmatrix} x \\ x \\ y \end{bmatrix}^\top N \begin{bmatrix} x \\ x \\ y \end{bmatrix} \geqslant 0.$$

Clearly, $x \neq 0$ since $N_{22} < 0$. Define $W := \begin{bmatrix} I_n & 0_{n,m} \end{bmatrix}$ and

$$N_W := \begin{bmatrix} W^\top & 0 \\ 0 & I_{n+m} \end{bmatrix} N \begin{bmatrix} W & 0 \\ 0 & I_{n+m} \end{bmatrix}.$$

Then

$$\begin{bmatrix} x \\ y \\ x \\ y \end{bmatrix}^\top N_W \begin{bmatrix} x \\ y \\ x \\ y \end{bmatrix} \geqslant 0.$$

By Lemma A.15, there exists a matrix $Z \in \mathcal{Z}_{n+m}(N_W)$ such that

$$\begin{bmatrix} x \\ y \\ x \\ y \end{bmatrix} = \begin{bmatrix} I_{n+m} \\ Z \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

so

$$\begin{bmatrix} x \\ y \end{bmatrix} = Z \begin{bmatrix} x \\ y \end{bmatrix}.$$

Moreover, by Theorem A.7, $Z = \bar{Z}W$ where $\bar{Z} \in \mathcal{Z}_{n+m}(N)$. Hence, $\begin{bmatrix} x \\ y \end{bmatrix} = \bar{Z}x$. Partition

$$\bar{Z} = \begin{bmatrix} \bar{A}^\top \\ \bar{B}^\top \end{bmatrix}.$$

Then $\bar{A}^\top x = x$. Because the data are informative for stability, $\bar{A}^\top$ is stable. However, since $x \neq 0$, $\bar{A}^\top$ also has an eigenvalue 1. We thus reach a contradiction. In other words, $\Psi(1) < 0$. This shows that item (c) holds.

Finally, we prove that (c) $\implies$ (a). Since $\Psi(1) < 0$ and the matrix (3.56) has no eigenvalues on the imaginary axis, by Lemma 3.26, $\Psi(\lambda) < 0$ for all $\lambda \in \mathcal{C}$. Therefore, by applying Proposition 3.27 with $A = 0$ and $B = \begin{bmatrix} I_n & 0_{n,m} \end{bmatrix}$ we conclude that the matrix inequality

$$\begin{bmatrix} P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & 0 \end{bmatrix} - N > 0 \tag{3.58}$$

has a solution $P \in \mathbb{S}^n$. To prove that $P > 0$ we note that $-N_{22}^{-1}N_{21} \in \mathcal{Z}_{n+m}(N)$. Partition $-N_{22}^{-1}N_{21} = \begin{bmatrix} A & B \end{bmatrix}^\top$ and note that $A^\top$ is stable by assumption. By multiplying (3.58) from the left by $\begin{bmatrix} I & A & B \end{bmatrix}$ and from the right by its transposed we obtain $P - APA^\top > 0$, that is, P satisfies the Lyapunov equation $P - APA^\top = Y$ for some matrix $Y > 0$. Since A is stable by assumption, the solution P is unique and given by $P = \sum_{k=0}^{\infty} A^k Y (A^\top)^k$. We observe that $P > 0$ because $Y > 0$. Therefore, we conclude that there exists a $P > 0$ satisfying (3.58), By Theorem 3.18 then, the data (U_-, X) are informative for quadratic stability. This proves the theorem. □

3.9.3 Stabilizability from noisy input-state data

To conclude this section, we turn our attention to data (U_-, X) obtained from the noisy input-state system (3.16) and study informativity for stabilizability as defined in Definition 3.13 (c). Recall that the condition for informativity for quadratic stabilizability in Corollary 3.20 is the same as the condition for informativity for quadratic stability of the state data X given in Theorem 3.24. One may thus think that an extension of Theorem 3.25 to stabilizability is straightforward. However, this extension is hindered by a subtle fact. To explain this, recall Lemma 3.30. Now, even though (3.53) holds, the subtlety is that the upper left $2n \times 2n$ submatrix of the matrix in (3.53) is generally not a member of $\mathbf{\Pi}_{n,n}$, i.e., (3.42) does not hold. Nonetheless, we provide the following theorem that characterizes informativity for stabilizability and quadratic stabilizability in the case that (3.42) is satisfied. Again, let the matrix Θ be as defined in (3.43).

Theorem 3.32. *Let the data (U_-, X) be obtained from (3.16) where the noise satisfies (3.18) with $\Phi \in \mathbf{\Pi}_{n,T}$ and $\Phi_{22} < 0$. Define the mapping Ψ as in (3.45). Assume that the data (U_-, X) satisfy (3.42) and the full rank condition (3.35). Suppose that $\Psi(1)$ is invertible and the matrix*

$$\begin{bmatrix} 0 & \Psi(1)^{-1} \\ \Psi(-1) & 2(\Theta X_-^\top - X_- \Theta^\top)\Psi(1)^{-1} \end{bmatrix} \tag{3.59}$$

has no eigenvalues on the imaginary axis. Then the following statements are equivalent:

(a) *(U_-, X) are informative for quadratic stabilizability.*

(b) *(U_-, X) are informative for stabilizability.*

(c) *$\Psi(1) < 0$ and the matrix $(X_- \Phi_{22} X_-^\top)^{-1} X_- \Theta^\top$ is stable.*

Proof. The implication (a) $\implies$ (b) is clear. Next, we prove that (b) $\implies$ (c). Let N be the matrix in (3.42) and N' be the matrix in (3.53). Since (3.42) holds by hypothesis, we have that

$$\begin{bmatrix} (X_-\Phi_{22}X_-^\top)^{-1}X_-\Theta^\top \\ 0 \end{bmatrix} \in \mathcal{Z}_{n+m}(N').$$

In other words, $(\Theta X_-^\top(X_-\Phi_{22}X_-^\top)^{-1}, 0) \in \Sigma_\mathcal{D}$. Since every system in $\Sigma_\mathcal{D}$ is stabilizable, we must have that the matrix $\Theta X_-^\top(X_-\Phi_{22}X_-^\top)^{-1}$ is stable. We thus conclude that its transposed $(X_-\Phi_{22}X_-^\top)^{-1}X_-\Theta^\top$ is also stable.

Next, we prove that $\Psi(1) < 0$. Suppose on the contrary that there exists a nonzero vector $x \in \mathbb{R}^n$ such that $x^\top\Psi(1)x \geqslant 0$, equivalently, (3.50) holds. By Lemma A.15 there exists a matrix $Z \in \mathcal{Z}_n(N)$ such that (3.51) holds. Since $Zx = x$, Z has an eigenvalue 1 and is thus not stable. However, this is a contradiction because $(Z^\top, 0) \in \Sigma_\mathcal{D}$. As such, we conclude that $\Psi(1) < 0$.

Finally, we prove that (c) $\implies$ (a). Since $\Psi(1) < 0$ and (3.46) has no imaginary eigenvalues, we conclude by Corollary 3.28 that there exists a $P \in \mathbb{S}^n$ satisfying (3.48) with N given in (3.42). It remains to be shown that $P > 0$. To this end, note that $Z := (X_-\Phi_{22}X_-^\top)^{-1}X_-\Theta^\top \in \mathcal{Z}_n(N)$. Hence, by multiplying (3.48) from the left by $\begin{bmatrix} I & Z^\top \end{bmatrix}$ and from the right by its transposed we obtain $P - Z^\top PZ > 0$, that is, P satisfies the Lyapunov equation $P - Z^\top PZ = Y$ for some matrix $Y > 0$. Since Z is stable by hypothesis, the solution P is unique and given by $P = \sum_{k=0}^{\infty}(Z^\top)^k YZ^k$. Clearly, $P > 0$ because $Y > 0$. Therefore, we conclude that there exists a $P > 0$ satisfying (3.48), with N given by (3.42). Thus, by Corollary 3.20, the data X are informative for quadratic stabilizability. This proves the theorem. □

3.10 Controllability from noisy data

This section deals with informativity of noisy data for controllability. In contrast to our tests for stability and stabilizability, we will only establish sufficient conditions, again in terms of feasibility of a linear matrix inequality. To start with, we first state a necessary and sufficient condition for controllability of stable input-state systems.

Lemma 3.33. *Assume that A is stable. Then the system (A, B) is controllable if and only if there exists a matrix $P > 0$ such that $P - APA^\top - BB^\top \leqslant 0$.*

Proof. To prove the 'if' statement, let $v \in \mathbb{C}^n$, be such that $v^*A = \lambda v^*$ and $v^*B = 0$. Then $(1 - |\lambda|^2)\ v^*Pv \leqslant 0$. Since, by stability, $|\lambda| < 1$ we must have $v^*Pv = 0$, so $v = 0$. The result then follows from the Hautus test.

Conversely, if A is stable and (A, B) is controllable then the controllability Gramian $P > 0$ satisfies $P - APA^\top - BB^\top = 0$. This completes the proof. □

Again consider the model class $\mathcal{M}$ of all noisy input-state systems given by (3.17) together with input-state data (U_-, X) on the time interval $[0, T]$. The possible matrices W_- of noise samples satisfy the quadratic inequality (3.18) for a given $\Phi \in \mathbf{\Pi}_{n,T}$. The set $\Sigma_{\mathcal{D}}$ of systems consistent with the data is given by (3.20), and we assume that it contains the true system $(A_{\text{true}}, B_{\text{true}})$.

In the remainder of this section it will be assumed that $\Phi_{22} < 0$ and that the data (U_-, X) satisfy the full rank condition (3.35). Thus we have $N_{22} < 0$, which in turn is equivalent to boundedness of $\Sigma_{\mathcal{D}} = \mathcal{Z}_{n+m}(N)$ (see Theorem A.5). Obviously, in that case there exists $\gamma > 0$ such that $AA^\top < \gamma I$ for all $(A, B) \in \Sigma_{\mathcal{D}}$, which implies that $\frac{1}{\sqrt{\gamma}} A$ is stable for all $(A, B) \in \Sigma_{\mathcal{D}}$.

Our definition of informativity for controllability requires all systems that are consistent with the data to be controllable.

Definition 3.34. The data (U_-, X) are called *informative for controllability* if every $(A, B) \in \Sigma_{\mathcal{D}}$ is controllable.

We will now derive sufficient conditions for informativity for controllability. The idea it to first compute a scaling factor $\gamma > 0$ such that $AA^\top - \gamma I < 0$ for all $(A, B) \in \Sigma_{\mathcal{D}}$. This can be done in the following way. Note that $AA^\top - \gamma I < 0$ can be written as

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} \gamma I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} > 0. \tag{3.60}$$

As a consequence, we need to find $\gamma > 0$ such that the strict quadratic matrix inequality (3.60) holds for all (A, B) that satisfy the quadratic matrix inequality (3.27). For given $\gamma > 0$, define

$$M_1 := \begin{bmatrix} \gamma I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{3.61}$$

Again, let N be given by (3.30). Then we have to find conditions in terms of the matrices M_1 and N such that

$$\mathcal{Z}_{n+m}(N) \subseteq \mathcal{Z}^+_{n+m}(M_1). \tag{3.62}$$

By virtue of Theorem A.20 the inclusion (3.62) holds if and only if there exists $\alpha \geqslant 0$ such that $M_1 - \alpha N > 0$, equivalently

$$\begin{bmatrix} \gamma I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & I \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top > 0.$$

As a consequence, a suitable γ can be found by solving this linear matrix inequality.

Next, after fixing the scaling factor γ obtained above, we want to find $P > 0$ such that the (nonstrict) inequality $\gamma P - APA^\top - BB^\top \leqslant 0$ holds for all $(A,B) \in \Sigma_\mathcal{D}$. This inequality can also be reformulated as a quadratic matrix inequality in A and B. Indeed, for given γ and $P > 0$, the inequality holds if and only if

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} -\gamma P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} \geqslant 0. \tag{3.63}$$

Now define

$$M_2 := \begin{bmatrix} -\gamma P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{3.64}$$

We have to find conditions in terms of the matrices M_2 and N such that

$$\mathcal{Z}_{n+m}(N) \subseteq \mathcal{Z}_{n+m}(M_2). \tag{3.65}$$

By Theorem A.17, the inclusion (3.65) holds if and only if there exists $\alpha \geqslant 0$ such that

$$M_2 - \alpha N \geqslant 0. \tag{3.66}$$

The following theorem then gives a sufficient condition for informativity for controllability.

Theorem 3.35. *Assume that $\Phi_{22} < 0$ and that the full rank condition (3.35) holds. Let $\gamma > 0$ be such that $AA^\top - \gamma I < 0$ for all $(A,B) \in \Sigma_\mathcal{D}$. Then the data (U_-, X) are informative for controllability if there exist an $n \times n$ matrix $P > 0$, and a scalar $\alpha \geqslant 0$ satisfying*

$$\begin{bmatrix} -\gamma P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & I \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top \geqslant 0. \tag{3.67}$$

Proof. Suppose (3.67) holds for some $P > 0$ and $\alpha \geqslant 0$. Let $(A,B) \in \Sigma_\mathcal{D}$. By the inclusion (3.65) we then have $\gamma P - APA^\top - BB^\top \leqslant 0$. Since $\frac{1}{\sqrt{\gamma}}A$ is stable, this implies that $(\frac{1}{\sqrt{\gamma}}A, \frac{1}{\sqrt{\gamma}}B)$ is controllable, so (A,B) is controllable. $\square$

3.11 Notes and references

The notion of informativity for system identification and the characterization in Theorem 3.1 can be found in [175].

For the classical Hautus tests for controllability and stabilizability (see Equation 3.7) we refer to [160, Thms. 3.13 and 3.32].

The problem of verifying controllability of a linear system from input-state data has been studied in different papers. The authors of [182] consider m independent experiments, where the initial state of each experiment is zero and the inputs of the ith experiment are all equal to the ith standard basis vector of $\mathbb{R}^m$. Under these experimental conditions, they show how to verify controllability of the true system from the combined data. Extensions of the approach were considered in [94,119,199]. The data-driven Hautus tests for controllability and stabilizability in Theorem 3.3 were proven in [175]. One of the attractive features of these tests is that they can be applied to general sets of input-state data, that is, the inputs and initial states can take arbitrary values.

The results on informativity for stability are partially based on the paper [175]. In particular, [175] considered the problem of verifying stability of an autonomous system using state data (see Theorem 3.11). In this book, we have also studied the problem of verifying internal stability of an input-state system using input-state data. Stability analysis of autonomous linear systems has been considered before in [125]. In the notation of this book, the paper [125] assumes that X_- has full row rank. An interesting consequence of Theorem 3.11 is that this condition is *necessary* for informativity for stability. Data-driven stability analysis has also been considered for a class of switched linear systems [88] from a probabilistic viewpoint.

The energy bound on the noise in (3.18) has become a common modeling approach in recent work on data-driven control. In this particular form, (3.18) was introduced in [169]. However, related noise models have appeared elsewhere. In particular, [44] considers a special case of (3.18) while [18] makes use of a 'dual' noise model. Other references that work with energy bounded noise include [29, 89, 157]. We also refer to [168] for the discussion of special cases of the model (3.18).

In the setting of noisy data, the notions of informativity for stability, quadratic stability, stabilizability and quadratic stabilizability were introduced in [171], alongside their characterizations in Theorem 3.24, Corollary 3.20, and Theorems 3.25 and 3.32. In order to prove Theorems 3.25 and 3.32 we required some technical results, including Lemma 3.26 and Proposition 3.27. In the proof of Lemma 3.26 we make use of the fact that the largest eigenvalue of $\Psi(\lambda)$ is continuous, which follows from [87, p. 125-126]. In Example 3.29 we have studied a consensus protocol with a so-called stubborn agent. More details on such protocols can be found in [128].

4

Dissipativity analysis

In this chapter we consider data-driven dissipativity analysis. This problem is concerned with deciding on the basis of data whether an unknown input-state-output system is dissipative with respect to a given supply rate. We first consider the case that the input-state-output data obtained from the unknown system are noiseless. It will be shown that in this situation the data are informative for dissipativity if and only if they are informative for system identification and, in addition, the unique system consistent with the data is dissipative. Next, we turn to the noisy case. We consider an unknown input-state-output system corrupted by unknown process noise and measurement noise. The matrix collecting the samples of these noise signals are assumed to satisfy a given quadratic matrix inequality. For two different set-ups of these quadratic inequality constraints we establish necessary and sufficient conditions for informativity for dissipativity.

4.1 Dissipativity from noiseless data

Before turning to data-driven dissipativity analysis, we will first review the definition of dissipativity. Consider a discrete-time linear input-state-output system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{4.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$ are given matrices. Let $S \in \mathbb{S}^{m+p}$. The system (4.1) is said to be *dissipative* with respect to the *supply rate*

$$s(u,y) = \begin{bmatrix} u \\ y \end{bmatrix}^\top S \begin{bmatrix} u \\ y \end{bmatrix} \tag{4.2}$$

if there exists $P \in \mathbb{S}^n$ with $P \geqslant 0$ such that the *dissipation inequality*

$$x(t+1)^\top P x(t+1) - x(t)^\top P x(t) \leqslant s\big(u(t), y(t)\big) \tag{4.3}$$

holds for all $t \geqslant 0$ and for all trajectories $(u, x, y) : \mathbb{Z}_+ \to \mathbb{R}^{m+n+p}$ of (4.1). For any such matrix P, the function $x \mapsto x^\top P x$ is called a *storage function* for the system (4.1) and the supply rate (4.2). It follows from (4.3) that dissipativity with respect to the supply rate (4.2) is equivalent with the feasibility of the linear matrix inequalities $P \geqslant 0$ and

$$\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^\top S \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \geqslant 0. \tag{4.4}$$

In the framework of data-driven system analysis, the system matrices are unknown. The question we want to study then is whether we can verify dissipativity using only the input-state-output data obtained from the unknown system. In the present section we will study this question for the situation that our data are noiseless.

Consider the unknown input-state-output system

$$x(t+1) = A_{\text{true}}x(t) + B_{\text{true}}u(t) \tag{4.5a}$$
$$y(t) = C_{\text{true}}x(t) + D_{\text{true}}u(t) \tag{4.5b}$$

where the input u is m-dimensional, the state x is n-dimensional and the output y is p-dimensional. We assume that the dimensions m, n and p are known, but the true system matrices $(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})$ are unknown. What is known instead are a finite number of input-state-output measurements of (4.5).

More concrete, we suppose that we have collected input-state-output data on the time interval $[0, T]$. Let U_-, X, X_-, and X_+ be defined by (2.1) and (2.2) and let Y_- be defined in a similar way as U_-. Our data are now given by $\mathcal{D} = (U_-, X, Y_-)$. These data are assumed to be generated by the true system $(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})$, which means that

$$\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A_{\text{true}} & B_{\text{true}} \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}. \tag{4.6}$$

The set of all systems that are consistent with these data is then given by:

$$\Sigma_{(U_-,X,Y_-)} := \left\{ (A, B, C, D) \mid \begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\}. \tag{4.7}$$

It follows from (4.6) that the unknown system $(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})$ is contained in $\Sigma_{(U_-,X,Y_-)}$. Our goal is to infer from the data (U_-, X, Y_-) whether the unknown system (4.5) is dissipative.

On the basis of the given data we are unable to distinguish between the systems in $\Sigma_{(U_-,X,Y_-)}$, in the sense that any of these systems could have generated the data. Nonetheless, if all of these systems are dissipative, then we can also conclude that the true data-generating system (4.5) is dissipative. With this in mind, we now define the property of *informativity for dissipativity* for the case of noiseless data.

Definition 4.1. The data (U_-, X, Y_-) are *informative for dissipativity* with respect to the supply rate (4.2) if there exists a matrix $P \in \mathbb{S}^n$, $P \geqslant 0$, such that the LMI (4.4) holds for every system $(A, B, C, D) \in \Sigma_{(U_-,X,Y_-)}$.

Note that our definition of informativity for dissipativity requires the systems in $\Sigma_{(U_-,X,Y_-)}$ to be dissipative with a *common* storage function.

We will restrict ourselves to the case that the number of negative eigenvalues of the matrix S representing the supply rate is equal to the output dimension p and the number of positive eigenvalues of S is equal to the input dimension m. In particular then, S is nonsingular. In other words, we will impose the following assumption on the inertia of S:

$$\operatorname{In}(S) = (p, 0, m). \tag{4.8}$$

It is a well-known fact that a necessary condition for dissipativity of any system of the form (4.1) is that $m \leqslant \operatorname{In}_+(S)$, i.e., the input dimension does not exceed the positive signature of S. Our assumption requires that the input dimension is equal to this positive signature and in addition that the matrix S is nonsingular. This assumption is satisfied, for example, for the positive-real and bounded-real case. Indeed, in the positive-real case we have that $m = p$ and

$$S = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$$

so that $\operatorname{In}(S) = (m, 0, m)$. In the bounded-real case we have

$$S = \begin{bmatrix} \gamma^2 I_m & 0 \\ 0 & -I_p \end{bmatrix}$$

for some $\gamma > 0$, which implies that $\operatorname{In}(S) = (p, 0, m)$.

Before establishing conditions for informativity for dissipativity, recall the notion of informativity for identification (see Section 3.1). In particular, recall from Theorem 3.1 that the noiseless input-state data (U_-, X) are informative for system identification if and only if the full rank condition

$$\operatorname{rank} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = n + m \tag{4.9}$$

holds. This notion can of course be extended to noiseless input-state-output data. Indeed, in accordance with Definition 2.12, we define:

Definition 4.2. The data (U_-, X, Y_-) are informative for system identification if $\Sigma_{(U_-,X,Y_-)}$ contains exactly one element.

If this is the case, then by (4.6) this single element must coincide with the unknown system $(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})$. It turns out that the data (U_-, X, Y_-) are informative for system identification if and only if the corresponding input-state data (U_-, X) are informative.

Theorem 4.3. *The data (U_-, X, Y_-) are informative for system identification if and only if the rank condition (4.9) holds. In that case we have $A_{\text{true}} = X_+ V_1$, $B_{\text{true}} = X_+ V_2$, $C_{\text{true}} = Y_- V_1$ and $D_{\text{true}} = Y_- V_2$, where V_1 and V_2 are such that*

$$\begin{bmatrix} X_- \\ U_- \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix}. \tag{4.10}$$

Proof. The set $\Sigma_{(U_-, X, Y_-)}$ of systems that are consistent with the data contains exactly one element if and only if the solution set of the homogeneous equation

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}$$

only contains $(0_{n,n}, 0_{n,m}, 0_{p,n}, 0_{p,m})$. This is the case if and only if (4.9) holds. For any right inverse $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ the unique solution $(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})$ of the inhomogeneous linear equation

$$\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}$$

is then given by

$$\begin{bmatrix} A_{\text{true}} & B_{\text{true}} \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix} = \begin{bmatrix} X_+ \\ Y_- \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}.$$

This completes the proof. ☐

As the main result of this section we will now show that the noiseless input-state-output data (U_-, X, Y_-) are informative for dissipativity if and only if they are informative for system identification and the unique system consistent with these data is dissipative. In addition, dissipativity of this unknown true system can be expressed in terms of feasibility of an LMI involving the data.

Theorem 4.4. *Assume that $\text{In}(S) = (p, 0, m)$. Then the data (U_-, X, Y_-) are informative for dissipativity with respect to the supply rate (4.2) if and only if they are informative for system identification and there exists $P \geqslant 0$ such that*

$$\begin{bmatrix} X_- \\ X_+ \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} X_- \\ X_+ \end{bmatrix} + \begin{bmatrix} U_- \\ Y_- \end{bmatrix}^\top S \begin{bmatrix} U_- \\ Y_- \end{bmatrix} \geqslant 0. \tag{4.11}$$

Proof. We first prove the 'only if' part. Suppose that the data are not informative for system identification. Then by Theorem 4.3 the rank condition (4.9) does not hold, so there exist $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$ such that $\xi^\top \xi + \eta^\top \eta = 1$ and

$$\begin{bmatrix} \xi^\top & \eta^\top \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = 0. \tag{4.12}$$

The set $\Gamma = \{u \in \mathbb{R}^m \mid \exists\, y \in \mathbb{R}^p \text{ such that } s(u,y) < 0\}$ has nonempty interior since there exists $(\hat{u}, \hat{y})$ with $s(\hat{u},\hat{y}) < 0$ due to our assumption on the inertia of S. We claim that there exist $x \in \mathbb{R}^n$ and $u \in \Gamma$ such that

$$\xi^\top x + \eta^\top u = 1. \tag{4.13}$$

Indeed, if $\xi \neq 0$, then one can construct x and u by selecting $u \in \Gamma$ arbitrarily, and by defining $x := \frac{1-\eta^\top u}{\xi^\top \xi}\xi$. If $\xi = 0$ then $x \in \mathbb{R}^n$ can be selected arbitrarily. In this case, we can choose u as follows. Since Γ has nonempty interior, there exists $\bar{u} \in \Gamma$ such that $\eta^\top \bar{u} \neq 0$. Note that $\alpha\bar{u} \in \Gamma$ for all nonzero $\alpha \in \mathbb{R}$. As such, there exists an $\alpha \in \mathbb{R}$ such that $u := \alpha\bar{u} \in \Gamma$ and $\eta^\top u = 1$. For this u, we obtain (4.13) which proves our claim.

Since $u \in \Gamma$, there exists y such that $s(u,y) < 0$. Let $(A_0, B_0, C_0, D_0) \in \Sigma_{(U_-,X,Y_-)}$. Define

$$\zeta := x - A_0 x - B_0 u \quad \text{and} \quad \theta := y - C_0 x - D_0 u \tag{4.14}$$

and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} := \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} + \begin{bmatrix} \zeta \\ \theta \end{bmatrix} \begin{bmatrix} \xi^\top & \eta^\top \end{bmatrix}.$$

It follows from (4.12) that $(A,B,C,D) \in \Sigma_{(U_-,X,Y_-)}$. Since the data are informative for dissipativity with respect to the supply rate (4.2), there must exist $P \geqslant 0$ such that the linear matrix inequality (4.4) holds. Note that

$$\begin{bmatrix} I & 0 \\ A & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} u \\ y \end{bmatrix}$$

due to (4.13) and (4.14). Therefore, the following inequality holds:

$$\begin{aligned} &\begin{bmatrix} x \\ u \end{bmatrix}^\top \left(\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^\top S \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \right) \begin{bmatrix} x \\ u \end{bmatrix} \\ &= \begin{bmatrix} x \\ x \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} u \\ y \end{bmatrix}^\top S \begin{bmatrix} u \\ y \end{bmatrix} = s(u,y) < 0. \end{aligned}$$

However, this contradicts (4.4). Consequently, the full rank condition (4.9) holds. Next, it then follows from Theorem 4.3 that

$$\Sigma_{(U_-,X,Y_-)} = \{(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})\}.$$

Define

$$L := \begin{bmatrix} I & 0 \\ A_{\text{true}} & B_{\text{true}} \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} I & 0 \\ A_{\text{true}} & B_{\text{true}} \end{bmatrix} + \begin{bmatrix} 0 & I \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix}^\top S \begin{bmatrix} 0 & I \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix}.$$

Since the data are informative for dissipativity, there exists $P = P^\top \geqslant 0$ such that $L \geqslant 0$. By post- and pre-multiplying this expression by $\begin{bmatrix} X_- \\ U_- \end{bmatrix}$ and its transpose, we conclude that (4.11) holds.

To prove the 'if' part, note that by assumption we have that (4.6) holds. Then (4.11) implies

$$\begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top L \begin{bmatrix} X_- \\ U_- \end{bmatrix} \geqslant 0. \tag{4.15}$$

It immediately follows from the full rank condition (4.9) that $L \geqslant 0$. By (4.4), this means that the system $(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})$ is dissipative with respect to the supply rate (4.2). Finally, since $\Sigma_{(U_-,X,Y_-)} = \{(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})\}$, we conclude that the data are informative for dissipativity. This completes the proof. □

Remark 4.5. The result of Theorem 4.4 implies that it is *only* possible to ascertain dissipativity from noise-free data if the plant is uniquely identifiable, in the sense that the data are informative for system identification. Consequently, in the noise-free setting, methods for determining dissipativity directly from data are conceptually equivalent with indirect ones consisting of a system identification stage, followed by a second one involving a check on the solvability of an LMI (condition (4.4)).

4.2 Dissipativity from noisy data

In this section we proceed with studying informativity for dissipativity in the case that our input-state-output data are obtained from an unknown system subject to unknown process noise and measurement noise. We assume that the unknown system is given by

$$x(t+1) = A_{\text{true}}x(t) + B_{\text{true}}u(t) + w(t) \tag{4.16a}$$

$$y(t) = C_{\text{true}}x(t) + D_{\text{true}}u(t) + z(t) \tag{4.16b}$$

where the input u is m-dimensional, the state x is n-dimensional and the output y is p-dimensional. The dimensions m, n and p are assumed to be known. The terms w and z are n-dimensional and p-dimensional, respectively. They represent process and measurement noise, respectively, and are assumed to be unknown. Also the system matrices $(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})$ are assumed to be unknown.

Again, we assume that a supply rate is represented by a given matrix $S \in \mathbb{S}^{m+p}$, viz. (4.2). The problem that we will study in this section is to determine whether the unknown system (4.16) is dissipative with respect to the given supply rate.

Suppose that we obtain input-state-output data data from the unknown system (4.16) on the time interval $[0, T]$. These data are collected in the matrices (U_-, X, Y_-) that are given as before by

$$\begin{aligned} U_- &= U_{[0,T-1]} \\ X &= X_{[0,T]} \\ Y_- &= Y_{[0,T-1]}. \end{aligned}$$

Also the auxiliary matrices X_- and X_+ are as defined before. The noise terms w and z are unknown, so $w(0), w(1), \ldots, w(T-1)$ and $z(0), z(1), \ldots, z(T-1)$ are not measured, and are therefore not part of the data. We denote

$$\begin{aligned} W_- &= W_{[0,T-1]} \\ Z_- &= Z_{[0,T-1]}. \end{aligned}$$

As part of the data $\mathcal{D}$ we do assume that we have the following information on the noise during the data sampling period.

Assumption 4.6. The noise samples, collected in the real $(n+p) \times T$ matrix

$$V_- := \begin{bmatrix} W_- \\ Z_- \end{bmatrix}$$

satisfy the quadratic matrix inequality

$$\begin{bmatrix} I \\ V_-^\top \end{bmatrix}^\top \Phi \begin{bmatrix} I \\ V_-^\top \end{bmatrix} \geqslant 0 \tag{4.19}$$

where $\Phi \in \mathbb{S}^{n+p+T}$ is a given partitioned matrix

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \tag{4.20}$$

with $\Phi_{11} \in \mathbb{S}^{n+p}$, $\Phi_{12} \in \mathbb{R}^{(n+p)\times T}$, $\Phi_{21} = \Phi_{12}^\top$ and $\Phi_{22} \in \mathbb{S}^T$. We assume that $\Phi \in \mathbf{\Pi}_{n+p,T}$. Then $\mathcal{Z}_T(\Phi)$ is nonempty and convex (see Theorem A.5). Moreover, V_- satisfies (4.19) if and only if $V_-^\top \in \mathcal{Z}_T(\Phi)$ (see Section A.2).

In other words, the data $\mathcal{D}$ consist of the input-state-output measurements (U_-, X, Y_-) together with the information that the noise on the sampling interval $[0, T]$ satisfies the inequality (4.19) for a given partitioned matrix Φ with the properties stated above.

We now turn to defining the property of *informativity for dissipativity* for noisy input-state-output data, i.e. data that are generated by the unknown system (4.16) with unknown process noise and measurement noise whose samples satisfy the quadratic matrix inequality (4.19). For our model class $\mathcal{M}$ we take all noisy input-state-output systems

$$x(t+1) = Ax(t) + Bu(t) + w(t) \tag{4.21a}$$
$$y(t) = Cx(t) + Du(t) + z(t) \tag{4.21b}$$

with input dimension m, state space dimension n and output dimension p. Given the input-state-output data (U_-, X, Y_-) together with the information that the matrices of noise samples satisfy (4.19), the set of all systems consistent with the data is then given by

$$\Sigma_{\mathcal{D}} = \left\{ (A,B,C,D) \mid \left(\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right)^\top \in \mathcal{Z}_T(\Phi) \right\}. \tag{4.22}$$

We assume that the data have been obtained from the unknown system (4.16), i.e., $(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}}) \in \Sigma_{\mathcal{D}}$. Therefore, $\Sigma_{\mathcal{D}}$ is nonempty. Define

$$N := \left[\begin{array}{c:c} N_{11} & N_{12} \\ \hdashline N_{12}^\top & N_{22} \end{array}\right] = \left[\begin{array}{c:c} I & \begin{matrix} X_+ \\ Y_- \end{matrix} \\ \hdashline 0 & \begin{matrix} -X_- \\ -U_- \end{matrix} \end{array}\right] \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \left[\begin{array}{c:c} I & \begin{matrix} X_+ \\ Y_- \end{matrix} \\ \hdashline 0 & \begin{matrix} -X_- \\ -U_- \end{matrix} \end{array}\right]^\top \tag{4.23}$$

Note that $(A,B,C,D) \in \Sigma_{\mathcal{D}}$ if and only if

$$\left[\begin{array}{cc} \multicolumn{2}{c}{I} \\ \hdashline A^\top & C^\top \\ B^\top & D^\top \end{array}\right]^\top N \left[\begin{array}{cc} \multicolumn{2}{c}{I} \\ \hdashline A^\top & C^\top \\ B^\top & D^\top \end{array}\right] \geqslant 0. \tag{4.24}$$

This can be restated equivalently as

$$\begin{bmatrix} A^\top & C^\top \\ B^\top & D^\top \end{bmatrix} \in \mathcal{Z}_{n+m}(N).$$

From Assumption 4.6 we have $\Phi_{22} \leqslant 0$ and therefore $N_{22} \leqslant 0$. It follows from the assumption $\ker \Phi_{22} \subseteq \ker \Phi_{12}$ that $\ker N_{22} \subseteq \ker N_{12}$. Since $\mathcal{Z}_{n+m}(N)$ is nonempty, it follows from inequality (A.10) that $N \mid N_{22} \geqslant 0$. Thus the matrix N given by (4.23) is in $\mathbf{\Pi}_{n+p,n+m}$.

Next, we give the definition of informativity for dissipativity in the context of noisy input-state-output data. Again, we will require that all systems consistent with the data are dissipative with a common storage function.

Definition 4.7. The noisy input-state-output data (U_-, X, Y_-) are *informative for dissipativity* with respect to the supply rate (4.2) if there exists a matrix $P \geqslant 0$ such that the LMI (4.4) holds for all systems $(A, B, C, D) \in \Sigma_{\mathcal{D}}$.

Similar to the noiseless case as studied in Section 4.1, in the remainder of this section we will assume that the matrix S representing the supply rate satisfies the inertia condition $\operatorname{In}(S) = (p, 0, m)$.

The following preliminary lemma states that also in the context of noisy data, the rank condition (4.9) on the input-state data is necessary for informativity.

Lemma 4.8. *Assume that* $\operatorname{In}(S) = (p, 0, m)$*. If the data* (U_-, X, Y_-) *are informative for dissipativity with respect to the supply rate* (4.2) *then*

$$\operatorname{rank} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = n + m. \tag{4.25}$$

Proof. Suppose that (4.25) does not hold, i.e., there exist $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$ such that $\xi^\top \xi + \eta^\top \eta = 1$ and

$$\begin{bmatrix} \xi^\top & \eta^\top \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = 0. \tag{4.26}$$

The set $\Gamma = \{u \in \mathbb{R}^m \mid \exists\, y \in \mathbb{R}^p \text{ such that } s(u, y) < 0\}$ has nonempty interior since there exists $(\hat{u}, \hat{y})$ with $s(\hat{u}, \hat{y}) < 0$. Similar as in the proof of Theorem 4.4, there exist $x \in \mathbb{R}^n$ and $u \in \Gamma$ such that

$$\xi^\top x + \eta^\top u = 1. \tag{4.27}$$

Since $u \in \Gamma$, there exists y such that $s(u, y) < 0$. Let $(A_0, B_0, C_0, D_0) \in \Sigma_{\mathcal{D}}$, equivalently

$$\left(\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} - \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right)^\top \in \mathcal{Z}_T(\Phi).$$

Define

$$\zeta := x - A_0 x - B_0 u \quad \text{and} \quad \theta := y - C_0 x - D_0 u \tag{4.28}$$

and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} := \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} + \begin{bmatrix} \zeta \\ \theta \end{bmatrix} \begin{bmatrix} \xi^\top & \eta^\top \end{bmatrix}.$$

It follows from (4.26) that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}$$

and therefore $(A, B, C, D) \in \Sigma_{\mathcal{D}}$ as well. Since the data are informative for dissipativity with respect to the supply rate (4.2), there exists $P \geqslant 0$ such that the dissipation inequality (4.4) holds. As before, due to (4.27) and (4.28) we have

$$\begin{bmatrix} I & 0 \\ A & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} u \\ y \end{bmatrix}.$$

Therefore, the following inequality holds:

$$\begin{aligned} &\begin{bmatrix} x \\ u \end{bmatrix}^\top \left(\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^\top S \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \right) \begin{bmatrix} x \\ u \end{bmatrix} \\ &= \begin{bmatrix} x \\ x \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} u \\ y \end{bmatrix}^\top S \begin{bmatrix} u \\ y \end{bmatrix} = s(u,y) < 0\,. \end{aligned}$$

However, this contradicts (4.4). Consequently, the full rank condition (4.25) holds. ☐

The following lemma states that if the data are informative for dissipativity with all systems in $\Sigma_{\mathcal{D}}$ having a common storage function $P \geqslant 0$, then P is necessarily *positive definite*. We will prove this under the additional assumption that the Schur complement $N \mid N_{22}$ is positive definite. Combining this with the fact that $N \in \mathbf{\Pi}_{n+p,n+m}$ as was already established above, we see from Theorem A.5 that $\Sigma_{\mathcal{D}}$ has a nonempty interior. The positive definiteness of P will play an important role in the remainder of the chapter, as it will be instrumental in deriving LMI conditions under which the data are informative for dissipativity.

Lemma 4.9. *Suppose that* $\mathrm{In}(S) = (p, 0, m)$ *and that* $N \mid N_{22} > 0$. *If* $P \geqslant 0$ *satisfies the dissipation inequality* (4.4) *for all* $(A, B, C, D) \in \Sigma_{\mathcal{D}}$ *then* $P > 0$.

Before we can prove Lemma 4.9, we need the following technical lemma on the inertia of certain products of matrices. Again let $\mathrm{In}_+(M)$ and $\mathrm{In}_-(M)$ denote the number of positive and negative eigenvalues of a symmetric matrix M. In addition, let $\dim \mathcal{V}$ denote the dimension of a subspace $\mathcal{V} \subseteq \mathbb{R}^n$.

Lemma 4.10. *Let* $M \in \mathbb{S}^n$ *and* $H \in \mathbb{R}^{n \times m}$. *Then*

$$\mathrm{In}_+(M) + (m - n) - \dim\ker H \leqslant \mathrm{In}_+(H^\top M H)$$

and

$$\mathrm{In}_-(M) + (m - n) - \dim\ker H \leqslant \mathrm{In}_-(H^\top M H).$$

Proof. We will first prove that

$$\mathrm{In}_+(H^\top M H) \geqslant \mathrm{In}_+(M) + \mathrm{In}_+(H^\top H) - n.$$

Denote $r := \mathrm{In}_+(H^\top H)$ and $s := \mathrm{In}_+(M)$. Then there exists an r-dimensional subspace $\mathcal{V} \subseteq \mathbb{R}^m$ such that $v^\top H^\top H v > 0$ for all nonzero $v \in \mathcal{V}$. Also, there exists an s-dimensional subspace $\mathcal{W} \subseteq \mathbb{R}^n$ such that $w^\top M w > 0$ for all nonzero $w \in \mathcal{W}$. Note that $\mathcal{V} \cap \ker H = \{0\}$ and therefore $\dim H\mathcal{V} = r$. Define a subspace $\mathcal{U} \subseteq \mathcal{V}$ by

$$\mathcal{U} := \{v \in \mathcal{V} \mid Hv \in \mathcal{W}\}.$$

Then obviously $v^\top H^\top M H v > 0$ for all nonzero $v \in \mathcal{U}$. This implies

$$\mathrm{In}_+(H^\top M H) \geqslant \dim \mathcal{U}.$$

Also, by the fact that $\mathcal{V} \cap \ker H = \{0\}$, we have $\dim \mathcal{U} = \dim(H\mathcal{V} \cap \mathcal{W})$. Using the fact that

$$\dim(H\mathcal{V} \cap \mathcal{W}) = \dim H\mathcal{V} + \dim \mathcal{W} - \dim(H\mathcal{V} + \mathcal{W})$$

together with $H\mathcal{V} + \mathcal{W} \subseteq \mathbb{R}^n$, we then obtain $\mathrm{In}_+(H^\top M H) \geqslant \dim \mathcal{U} \geqslant r + s - n$, as claimed. The proof of the first statement of the lemma is then completed by noting that

$$r = \mathrm{In}_+(H^\top H) = m - \dim \ker H^\top H = m - \dim \ker H.$$

To prove the second statement, note that the number of positive and negative eigenvalues are interchanged by replacing M by $-M$ and $H^\top M H$ by $-H^\top M H = H^\top(-M)H$. $\square$

Proof of Lemma 4.9. Let $\xi \in \ker P$. It follows from (4.4) that

$$-\begin{bmatrix} \xi^\top A^\top \\ B^\top \end{bmatrix} P \begin{bmatrix} A\xi & B \end{bmatrix} + \begin{bmatrix} 0 & I \\ C\xi & D \end{bmatrix}^\top S \begin{bmatrix} 0 & I \\ C\xi & D \end{bmatrix} \geqslant 0$$

for all $(A, B, C, D) \in \Sigma_{\mathcal{D}}$. This implies that

$$\begin{bmatrix} 0 & I \\ C\xi & D \end{bmatrix}^\top S \begin{bmatrix} 0 & I \\ C\xi & D \end{bmatrix} \geqslant 0$$

for every $(A, B, C, D) \in \Sigma_{\mathcal{D}}$. It now immediately follows from Lemma 4.10 that

$$\mathrm{In}_-(S) + (m + 1) - (m + p) - \dim \left(\ker \begin{bmatrix} 0 & I \\ C\xi & D \end{bmatrix} \right) \leqslant 0.$$

Using the assumption that $\mathrm{In}_-(S) = p$ this yields

$$\dim \left(\ker \begin{bmatrix} 0 & I \\ C\xi & D \end{bmatrix} \right) \geqslant 1.$$

Therefore, $C\xi = 0$ for every $(A, B, C, D) \in \Sigma_{\mathcal{D}}$. Moreover, since $N \,|\, N_{22} > 0$ by assumption, it follows from Theorem A.5 that $\Sigma_{\mathcal{D}}$ has a nonempty interior. As a consequence we have $C\xi = 0$ for a sufficiently rich set of matrices C so that we can conclude that $\xi = 0$. This implies that $P > 0$. □

Our next step is to partition

$$S = \begin{bmatrix} F & G \\ G^\top & H \end{bmatrix} \tag{4.29}$$

where $F \in \mathbb{R}^{m\times m}$, $G \in \mathbb{R}^{m\times p}$, $H \in \mathbb{R}^{p\times p}$. For any $P \geqslant 0$ define

$$M := \begin{bmatrix} P & 0 & 0 & 0 \\ 0 & F & 0 & G \\ 0 & 0 & -P & 0 \\ 0 & G^\top & 0 & H \end{bmatrix}. \tag{4.30}$$

Then the system (A, B, C, D) can be seen to satisfy the dissipation inequality (4.4) if and only if

$$\begin{bmatrix} I \\ \hdashline A \;\; B \\ C \;\; D \end{bmatrix}^\top M \begin{bmatrix} I \\ \hdashline A \;\; B \\ C \;\; D \end{bmatrix} \geqslant 0 \tag{4.31}$$

Moreover, with this notation in place, the problem of characterizing informativity for dissipativity is equivalent to finding conditions for the existence of a matrix $P > 0$ such that the inequality (4.31) holds for all (A, B, C, D) satisfying the inequality (4.24).

Our strategy to solve this problem is to invoke the nonstrict matrix S-lemma, Theorem A.17. Before we can apply Theorem A.17, however, note that the inequality (4.31) is in terms of (A, B, C, D) while the inequality (4.24) is in terms of the *transposed* matrices $(A^\top, C^\top, B^\top, D^\top)$. Therefore, we will need an additional dualization result that we formulate in the following lemma.

Lemma 4.11. *Let $P > 0$ and let (A, B, C, D) be any system with input dimension m, state space dimension n and output dimension p. Assume that* $\mathrm{In}(S) = (p, 0, m)$. *Define*

$$\hat{S} := \begin{bmatrix} 0 & -I_p \\ I_m & 0 \end{bmatrix} S^{-1} \begin{bmatrix} 0 & -I_m \\ I_p & 0 \end{bmatrix}. \tag{4.32}$$

Then we have

$$\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^\top S \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \geqslant 0 \tag{4.33}$$

if and only if

$$\begin{bmatrix} I & 0 \\ A^\top & C^\top \end{bmatrix}^\top \begin{bmatrix} P^{-1} & 0 \\ 0 & -P^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ A^\top & C^\top \end{bmatrix} + \begin{bmatrix} 0 & I \\ B^\top & D^\top \end{bmatrix}^\top \hat{S} \begin{bmatrix} 0 & I \\ B^\top & D^\top \end{bmatrix} \geqslant 0. \tag{4.34}$$

Proof. Partition the matrix S as in (4.29). Since P is positive definite and $\text{In}(S) = (p, 0, m)$, the inertia of the matrix M given by (4.30) is given by $\text{In}(M) = (n+p, 0, n+m)$. The lemma now readily follows by applying Lemma A.3 to the matrix M. □

Lemma 4.11 can be interpreted as saying that the system defined by the quadruple (A, B, C, D) is dissipative with respect to the supply rate S, with storage function P if and only if the dual system $(A^\top, C^\top, B^\top, D^\top)$ is dissipative with respect to the supply rate $\hat{S}$, with storage function P^{-1}.

Partition now

$$-S^{-1} = \begin{bmatrix} \hat{F} & \hat{G} \\ \hat{G}^\top & \hat{H} \end{bmatrix}$$

where $\hat{F} = \hat{F}^\top \in \mathbb{R}^{m \times m}$, $\hat{G} \in \mathbb{R}^{m \times p}$, and $\hat{H} = \hat{H}^\top \in \mathbb{R}^{p \times p}$ and define

$$\hat{M} := \begin{bmatrix} P^{-1} & 0 & 0 & 0 \\ 0 & \hat{H} & 0 & -\hat{G}^\top \\ 0 & 0 & -P^{-1} & 0 \\ 0 & -\hat{G} & 0 & \hat{F} \end{bmatrix}. \tag{4.35}$$

Then it is easily seen that $(A^\top, C^\top, B^\top, D^\top)$ satisfies the inequality (4.34) if and only if

$$\begin{bmatrix} I \\ \hdashline A^\top & C^\top \\ B^\top & D^\top \end{bmatrix}^\top \hat{M} \begin{bmatrix} I \\ \hdashline A^\top & C^\top \\ B^\top & D^\top \end{bmatrix} \geqslant 0. \tag{4.36}$$

We may now observe that, under the assumptions that $\text{In}(S) = (p, 0, m)$ and $N \,|\, N_{22} > 0$, informativity for dissipativity with respect to the supply rate given by S holds if and only if there exists $P > 0$ such that the quadratic inequality (4.36) holds for all (A, B, C, D) that satisfy the the quadratic inequality (4.24), equivalently

$$\mathcal{Z}_{n+m}(N) \subseteq \mathcal{Z}_{n+m}(\hat{M}). \tag{4.37}$$

This brings us in position to apply Theorem A.17 and to obtain the following characterization for informativity for dissipativity for noisy input-state-output data.

Theorem 4.12. *Assume that* $\text{In}(S) = (p, 0, m)$ *and that the data* (U_-, X, Y_-) *are such that* $N \,|\, N_{22} > 0$. *Partition*

$$-S^{-1} = \begin{bmatrix} \hat{F} & \hat{G} \\ \hat{G}^\top & \hat{H} \end{bmatrix} \tag{4.38}$$

where $\hat{F} = \hat{F}^\top \in \mathbb{R}^{m \times m}$, $\hat{G} \in \mathbb{R}^{m \times p}$, *and* $\hat{H} = \hat{H}^\top \in \mathbb{R}^{p \times p}$. *Then the data are informative for dissipativity with respect to the supply rate* (4.2) *if and only if there exist a real* $n \times n$ *matrix* $Q > 0$ *and a scalar* $\alpha \geqslant 0$ *such that*

$$\begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & \hat{H} & 0 & -\hat{G}^\top \\ 0 & 0 & -Q & 0 \\ 0 & -\hat{G} & 0 & \hat{F} \end{bmatrix} - \alpha \left[\begin{array}{c:c} I & X_+ \\ & Y_- \\ \hdashline 0 & -X_- \\ & -U_- \end{array}\right] \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \left[\begin{array}{c:c} I & X_+ \\ & Y_- \\ \hdashline 0 & -X_- \\ & -U_- \end{array}\right]^\top \geqslant 0. \tag{4.39}$$

In that case $P := Q^{-1}$ *is a common storage function for all systems consistent with the data.*

Proof. To prove the 'if' statement, assume that the LMI (4.39) holds for some $Q > 0$. Define $P := Q^{-1}$ and define $\hat{M}$ by (4.35). Then clearly $\mathcal{Z}_{n+m}(N) \subseteq \mathcal{Z}_{n+m}(\hat{M})$, so P is a common storage function for all $(A, B, C, D) \in \Sigma_{\mathcal{D}}$, which implies that we have informativity for dissipativity with respect to the supply rate (4.2).

To prove the 'only if' part, suppose that the data (U_-, X, Y_-) are informative for dissipativity. Using the assumptions $\text{In}(S) = (p, 0, m)$ and $N \,|\, N_{22} > 0$, this is equivalent with the existence of a matrix $P > 0$ such that the inclusion (4.37) holds, with $\hat{M}$ given by (4.35). Also recall that $N \in \mathbf{\Pi}_{n+p,n+m}$. In addition, N has at least one positive eigenvalue. Now define $Q := P^{-1}$. Then by Theorem A.17 there exists a scalar $\alpha \geqslant 0$ such that (4.39) holds. This completes the proof. ☐

Theorem 4.12 provides a tractable method for verifying informativity for dissipativity of noisy data given the noise model introduced in Assumption 4.6. The procedure involves solving the linear matrix inequality (4.39) for Q and α. Given Q, a common storage function P for all systems in $\Sigma_{\mathcal{D}}$ is also readily computable as $P = Q^{-1}$.

4.3 Informativity with an alternative noise model

In the preceding two sections we have studied informativity of input-state-data in two different setups. In Section 4.1 we assumed that our data are noiseless, and in Section 4.2 we considered the situation that samples of the process noise and measurement noise on a finite interval satisfy a given quadratic matrix inequality. This quadratic inequality was introduced in Assumption 4.6 and involved an a priori given weighting matrix Φ. In the present section we will discuss an alternative noise model and establish conditions for informativity of input-state-output data in the context of this alternative noise description.

Again our unknown system is assumed to be of the form (4.16). The input dimension m, state space dimension n and output dimension p are given. We also assume that a supply rate is given by a given matrix $S \in \mathbb{S}^{m+p}$, viz. (4.2). We have input, state and output samples U_-, X and Y_- and our aim is to determine on the basis of these data whether the unknown system is dissipative with respect to this supply rate. Again, the matrices A_{true}, B_{true}, C_{true} and D_{true} are unknown. Also the process noise w and measurement noise z are unknown. We do however assume that we have the following information on the possible noise samples on a given finite time interval.

Assumption 4.13. The noise samples, collected in the real $(n+p) \times T$ matrix

$$V_- := \begin{bmatrix} W_- \\ Z_- \end{bmatrix}$$

satisfy the quadratic matrix inequality

$$\begin{bmatrix} I \\ V_- \end{bmatrix}^\top \Theta \begin{bmatrix} I \\ V_- \end{bmatrix} \geqslant 0 \tag{4.40}$$

where $\Theta \in \mathbb{S}^{n+p+T}$ is a given partitioned matrix

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$$

with $\Theta_{11} \in \mathbb{S}^T$, $\Theta_{12} \in \mathbb{R}^{T \times (n+p)}$, $\Theta_{21} = \Theta_{12}^\top$ and $\Theta_{22} \in \mathbb{S}^{n+p}$. We assume that

$$\Theta_{22} < 0$$

and the Schur complement satisfies

$$\Theta \,|\, \Theta_{22} > 0.$$

Assumption 4.13 implies that $\Theta \in \mathbf{\Pi}_{T,n+p}$. In view of Theorem A.5, $\mathcal{Z}_{n+p}(\Theta)$ is bounded and has nonempty interior. Furthermore, V_- satisfies (4.40) if and only if $V_- \in \mathcal{Z}_{n+p}(\Theta)$ (see Section A.2).

Now, given the input-state-output data (U_-, X, Y_-) together with the information that the matrices of noise samples satisfy (4.40), the set of all systems consistent with the data is given by

$$\tilde{\Sigma}_{\mathcal{D}} = \left\{ (A,B,C,D) \mid \begin{bmatrix} X_+ \\ Y_- \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \in \mathcal{Z}_{n+p}(\Theta) \right\}. \tag{4.41}$$

The corresponding definition of informativity is then the following.

Definition 4.14. Given the noise model of Assumption 4.13, the noisy input-state-output data (U_-, X, Y_-) are *informative for dissipativity* with respect to the supply rate (4.2) if there exists a matrix $P \geqslant 0$ such that the LMI (4.4) holds for all systems $(A, B, C, D) \in \tilde{\Sigma}_{\mathcal{D}}$.

In order to obtain conditions for informativity in this new noise set up, we prove the following duality result.

Lemma 4.15. *Assume that $\Theta_{22} < 0$ and $\Theta \,|\, \Theta_{22} > 0$, equivalently, $\mathcal{Z}_{n+p}(\Theta)$ is bounded and has a nonempty interior. Define*

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} := \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}^{-1} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}. \tag{4.42}$$

Then for any $V \in \mathbb{R}^{(n+p)\times T}$ we have

$$V \in \mathcal{Z}_{n+p}(\Theta) \iff V^\top \in \mathcal{Z}_T(\Phi).$$

Proof. Note that $\operatorname{In}(\Theta) = (n+p, 0, T)$. The result then follows immediately from Lemma A.3. ☐

An immediate consequence of Lemma 4.15 is that a given system (A, B, C, D) is consistent with the data (U_-, X, Y_-) using the alternative noise model of Assumption 4.13 if and only if it is consistent with these data using the original noise model of Assumption 4.6 with Φ defined by (4.42). In other words,

$$\tilde{\Sigma}_{\mathcal{D}} = \left\{ (A, B, C, D) \mid \left(\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right)^\top \in \mathcal{Z}_T(\Phi) \right\}. \tag{4.43}$$

We now turn to finding conditions for informativity using the alternative noise model. In order to do this, we first need to check whether the matrix Φ defined by (4.42) satisfies the conditions that were imposed in Assumption 4.6. We will show that, in fact, $\Phi_{22} < 0$ and $\Phi \mid \Phi_{22} > 0$. Using the Schur decomposition (A.8) applied to the partitioned matrix Θ, it is easily seen that, in fact,

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} = \begin{bmatrix} -\Theta_{22}^{-1}\Theta_{21}(\Theta \,|\, \Theta_{22})^{-1}\Theta_{12}\Theta_{22}^{-1} & -\Theta_{22}^{-1}\Theta_{21}(\Theta \,|\, \Theta_{22})^{-1} \\ -(\Theta \,|\, \Theta_{22})^{-1}\Theta_{12}\Theta_{22}^{-1} & -(\Theta \,|\, \Theta_{22})^{-1} \end{bmatrix}. \tag{4.44}$$

From this we immediately seen that $\Phi_{22} = -(\Theta \mid \Theta_{22})^{-1} < 0$. It also follows from (4.42) that $\operatorname{In}(\Theta) = -\operatorname{In}(\Phi) = (T, 0, n+p)$. Since Φ_{22} is a $T \times T$ matrix, this implies that $\Phi \,|\, \Phi_{22} > 0$ as desired.

As before, in the sequel we will assume that the supply rate satisfies the inertia condition $\operatorname{In}(S) = (p, 0, m)$. Define

$$\tilde{N} := \left[\begin{array}{c:c} \tilde{N}_{11} & \tilde{N}_{12} \\ \hdashline \tilde{N}_{12}^\top & \tilde{N}_{22} \end{array}\right] = \left[\begin{array}{c:c} I & \begin{matrix} X_+ \\ Y_- \end{matrix} \\ \hdashline 0 & \begin{matrix} -X_- \\ -U_- \end{matrix} \end{array}\right] \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \left[\begin{array}{c:c} I & \begin{matrix} X_+ \\ Y_- \end{matrix} \\ \hdashline 0 & \begin{matrix} -X_- \\ -U_- \end{matrix} \end{array}\right]^\top$$

with Φ given by (4.44). For completeness we finally state the following characterization of informativity using the alternative noise model.

Theorem 4.16. *Assume that* $\operatorname{In}(S) = (p, 0, m)$ *and that the data* (U_-, X, Y_-) *are such that* $\tilde{N} \,|\, \tilde{N}_{22} > 0$*. Let* $-S^{-1}$ *be partitioned as in* (4.38)*. Then the data are informative for dissipativity with respect to the supply rate* (4.2) *using the noise model of Assumption 4.13 if and only if there exist a real* $n \times n$ *matrix* $Q > 0$ *and a scalar* $\alpha \geqslant 0$ *such that*

$$\begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & \hat{H} & 0 & -\hat{G}^\top \\ 0 & 0 & -Q & 0 \\ 0 & -\hat{G} & 0 & \hat{F} \end{bmatrix} - \alpha \left[\begin{array}{c:c} I & \begin{matrix} X_+ \\ Y_- \end{matrix} \\ \hdashline 0 & \begin{matrix} -X_- \\ -U_- \end{matrix} \end{array}\right] \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \left[\begin{array}{c:c} I & \begin{matrix} X_+ \\ Y_- \end{matrix} \\ \hdashline 0 & \begin{matrix} -X_- \\ -U_- \end{matrix} \end{array}\right]^\top \geqslant 0. \tag{4.45}$$

In that case $P := Q^{-1}$ *defines a common storage function for all systems consistent with the data.*

4.4 Notes and references

The notion of dissipativity was introduced by Jan Willems in the seminal papers [184] and [185]. During the same period, he also made fundamental contributions to the subject of optimal control, in particular to linear quadratic problems with indefinite cost, and the associated algebraic Riccati equation [183]. Together, [183], [184] and [185] are generally considered to provide the main concepts and analysis tools in many areas of linear and nonlinear systems and control, ranging from stability theory, linear quadratic optimal control and stochastic realization theory, to network synthesis, differential games and robust control. The above contributions addressed dissipativity of continuous-time systems. For an early contribution treating discrete-time systems, we refer to [31].

In this chapter, we have focused on the problem of verifying dissipativity of discrete-time LTI systems using measured data. The results that were presented here are based on the paper [170]. The problem of assessing dissipativity from data has received considerable attention in recent years, see e.g. [138] and [139].

In [107], the notion of (finite-horizon) L-dissipativity was introduced. This notion was also further studied in [137]. A discrete-time system is L-dissipative if the average of the supply rate over the time interval $[0, L]$ is nonnegative for all system trajectories. This is a necessary condition for dissipativity, but it is in general not sufficient. In both of the latter two contributions, a crucial assumption is that the input trajectory is persistently exciting of a sufficiently high order (see [190] and [172]). This property of the input sequence can be shown to imply that the data-generating system is uniquely identifiable from the data.

In the current chapter we have adopted the more classical notion of dissipativity for linear systems, rather than L-dissipativity. We have considered a setup similar to that of [89]. In that paper, sufficient data-based conditions were given for dissipativity. The main difference between our results and those in [89] is that we have provided necessary and sufficient conditions for dissipativity based on data, for noiseless and noisy data.

Apart from conditions for verifying dissipativity based on data, we have derived a number of additional results as byproducts that are interesting in their own right. First of all, we have shown in Corollary 4.15 that, under mild assumptions, the different noise models studied in the papers [18] and [169] are actually *equivalent*. Moreover, in the setting of noisy data, it follows from Lemma 4.9 that informativity for dissipativity requires the common storage function to be *positive definite*. This is an interesting conclusion, since the definition of dissipativity only requires positive semidefinite storage functions. We note that conditions under which all storage functions are positive definite have been studied before in [71, Lem. 1], for nonlinear systems. In that paper, certain minimality conditions were imposed as well as a signature condition on the supply rate. Here, we have not assumed minimality but we have concluded that all storage functions are positive definite in the case that the supply rate satisfies an inertia condition and the set of systems consistent with the data has nonempty interior. In order to prove Lemma 4.9, we have relied on Lemma 4.10 that can be interpreted as a generalization of Sylvester's law of inertia, see the paper [41].

5

Analysis of further system properties

In this chapter we will further study the problem of finding data-based tests for checking whether a given unknown dynamical system has certain structural properties. In Sections 3.2 and 3.3, tests were already established for checking whether a given set of noiseless data is informative for controllability, stabilizability and stability. For the case of noisy data, in Sections 3.5 to 3.10 data-driven tests were discussed for stability, stabilizability, quadratic stability, quadratic stabilizability and controllability. Also in this chapter, we will deal with informativity of noisy data. We will establish tests for informativity of several additional relevant structural system properties in the setting of unbounded noise. More specifically, we will study informativity for observability and detectability, strong observability and detectability, strong controllability and stabilizability, and invertibility of linear systems.

5.1 Problem setup

We will consider the linear input-state-output system with noise given by

$$x(t+1) = A_{\text{true}}x(t) + Bu(t) + Ew(t) \tag{5.1a}$$

$$y(t) = Cx(t) + Du(t) + Fw(t) \tag{5.1b}$$

where the state x is n-dimensional, the control input u is m-dimensional, the output y is p-dimensional, and the unknown noise w is r-dimensional. We assume that A_{true} is an unknown matrix, but the matrices B, C, D and E, F are known. The assumption that the matrix A_{true} is unknown while the others are known can, for example, be motivated within the context of networked systems, in which the input and output nodes are given, but the interconnection topology is unknown. Typically, in that context the matrices B, C and D are matrices whose columns only contain 0's and 1's, with in each column at most one entry equal to 1. The term Ew represents process noise, whereas Fw represents measurement noise. The special case that $E = 0$ and $F = 0$ is called the *noiseless case*.

We assume that we have input-state-output data concerning this unknown 'true' system in the form of samples of x, u and y on a given finite time interval

$[0, T]$. As before, these data are denoted by

$$\begin{aligned} U_- &= U_{[0,T-1]} \\ X &= X_{[0,T]} \\ Y_- &= Y_{[0,T-1]}. \end{aligned}$$

It will be assumed that these data are obtained from the true system (5.1), meaning that there exists some matrix

$$W_- = W_{[0,T-1]}$$

such that

$$X_+ = A_{\text{true}} X_- + BU_- + EW_- \tag{5.3a}$$

$$Y_- = CX_- + DU_- + FW_- \tag{5.3b}$$

where as usual we denote

$$\begin{aligned} X_- &= X_{[0,T-1]} \\ X_+ &= X_{[1,T]}. \end{aligned}$$

We then say that the data are consistent with the true system $(A_{\text{true}}, B, C, D, E, F)$.

The set of all $n \times n$ matrices A such that the system (A, B, C, D, E, F) is consistent with the data will be denoted by $\mathcal{A}_\mathcal{D}$, i.e.,

$$\mathcal{A}_\mathcal{D} := \left\{ A \in \mathbb{R}^{n \times n} \mid \exists W_- : \begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} + \begin{bmatrix} E \\ F \end{bmatrix} W_- \right\}. \tag{5.4}$$

Let $\mathcal{P}$ denote some system-theoretic property that might or might not hold for a given linear system. The general problem that we will address in this chapter is to determine from the data obtained from (5.1) whether the property $\mathcal{P}$ holds for the unknown true system $(A_{\text{true}}, B, C, D, E, F)$. Since on the basis of the data we can not distinguish between the true A_{true} and any $A \in \mathcal{A}_\mathcal{D}$, we need to check whether the property holds for *all* systems (A, B, C, D, E, F) with $A \in \mathcal{A}_\mathcal{D}$. In line with Definition 2.1 we then call the data informative for property $\mathcal{P}$.

Example 5.1. For $\mathcal{P}$ take the property '(A, B) is a controllable pair'. Suppose that on the basis of the data (U_-, X, Y_-) we want to determine whether $\mathcal{P}$ holds for the pair (A_{true}, B) corresponding to the true system. This requires to check whether the data are informative for property $\mathcal{P}$. Using ideas from Section 3.2, it can be shown that in the noiseless case (i.e. the case that $E = 0$ and $F = 0$) the data (U_-, X, Y_-) are informative for $\mathcal{P}$ if and only if rank $\begin{bmatrix} X_+ - \lambda X_- & B \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$. This will be proven later on in this chapter in Subsection 5.3.3. ■

Example 5.2. For $\mathcal{P}$ take the property 'the pair (C, A) is detectable'. In the noiseless case it can be shown that the data (U_-, X, Y_-) are informative for $\mathcal{P}$ if and only if $\ker C \subseteq \operatorname{im} X_-$ and for all $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$ we have

$$\operatorname{rank} \begin{bmatrix} X_+ - BU_- - \lambda X_- \\ CX_- \end{bmatrix} = \operatorname{rank} X_-.$$

This will be proven later on in this chapter in Subsection 5.3.2. ■

Remark 5.3. We note that the case of independent process noise and measurement noise is also covered by the noisy model (5.1) introduced above. The noise matrices should then be taken of the form $E = [E_1 \; 0]$ and $F = [0 \; F_2]$, while the noise signal is given by the vector

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

and, likewise,

$$W_- = \begin{bmatrix} W_{1-} \\ W_{2-} \end{bmatrix}.$$

A special case of this is that only process noise occurs, in which case F_2 is void and $E = E_1$ and $F = 0$. In other words, in the case of independent process and measurement noise we have $A \in \mathcal{A}_{\mathcal{D}}$ if and only if there exists a matrix W_{1-} such that $X_+ = AX_- + BU_- + E_1 W_{1-}$. The equation $Y_- = CX_- + DU_- + F_2 W_{2-}$ can then be ignored since it does not put any constraint on A.

This chapter will provide necessary and sufficient conditions on the input-state-output data obtained from (5.1) to be informative for a range of system properties $\mathcal{P}$. Throughout, we will restrict ourselves to the situation introduced above, namely, that the state matrix A_{true} is unknown, but that the matrices B, C and D are known. We will study both the noisy case as well as the noiseless case. In the noisy case it will be assumed that the noise matrices E and F are known.

The outline of this chapter is as follows. In Section 5.2, we will state and prove a theorem that will be instrumental in order to obtain our results on informativity in the rest of the chapter. This theorem states that a certain rank property of the system matrix of the unknown system is equivalent to a rank property of a polynomial matrix that collects available information about the unknown system. In Section 5.3, this result will be applied to obtain necessary and sufficient conditions for informativity of noisy data for the following system properties:

- strong observability and strong detectability of (A, B, C, D),

- observability and detectability of (C, A),
- strong controllability and strong stabilizability of (A, B, C, D),
- controllability and stabilizability of (A, B).

In Section 5.4, we apply ideas from the geometric approach to linear systems, to set up a geometric framework for informativity analysis for strong observability and observability. This framework will also be applied to the analysis of informativity for left-invertibility.

5.2 A rank property for an affine set of systems

In this section we will establish a general framework that will enable us to characterize informativity of input-state-output data for the properties listed in Section 5.1.

Let $P \in \mathbb{R}^{n\times r}$, $Q \in \mathbb{R}^{\ell\times n}$ and $R \in \mathbb{R}^{\ell\times r}$ be given matrices. Here, r and ℓ are positive integers, and the symbol n has the usual meaning of state space dimension. Using these matrices, we define an affine space of state matrices A by

$$\mathcal{A} := \{A \in \mathbb{R}^{n\times n} \mid R = QAP\}. \tag{5.5}$$

It is easily seen that $\mathcal{A}$ is nonempty if and only if the inclusions $\operatorname{im} R \subseteq \operatorname{im} Q$ and $\ker P \subseteq \ker R$ hold. Assume this to be the case.

Now let $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$ be given, and for each $A \in \mathcal{A}$ consider the system

$$x(t+1) = Ax(t) + Bu(t) \tag{5.6a}$$

$$y(t) = Cx(t) + Du(t). \tag{5.6b}$$

The system matrix associated with the system (5.6) is defined as the first order polynomial matrix

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}. \tag{5.7}$$

In addition, we will consider the polynomial matrix

$$\begin{bmatrix} R - sQP & QB \\ CP & D \end{bmatrix} \tag{5.8}$$

associated with the given matrices (P, Q, R) and (B, C, D). The following theorem expresses a uniform rank property of the set of system matrices (5.7), with A ranging over the affine set $\mathcal{A}$, in terms of a rank property of the single polynomial matrix (5.8).

Theorem 5.4. *Let (P, Q, R) and (B, C, D) be given. Then*

$$\operatorname{rank}\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} = n + \operatorname{rank}\begin{bmatrix} B \\ D \end{bmatrix} \tag{5.9}$$

for all $A \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ if and only if[1] $C^{-1}\operatorname{im} D \subseteq \operatorname{im} P$ *and*

$$\operatorname{rank}\begin{bmatrix} R-\lambda QP & QB \\ CP & D \end{bmatrix} = \operatorname{rank} P + \operatorname{rank}\begin{bmatrix} QB \\ D \end{bmatrix} \tag{5.10}$$

for all $\lambda \in \mathbb{C}$.

In addition, (5.9) holds for all $A \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$ if and only if $C^{-1}\operatorname{im} D \subseteq \operatorname{im} P$ and (5.10) holds for all $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$.

Proof. To start the proof, first observe that for any $A \in \mathcal{A}$:

$$\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix} = \begin{bmatrix} A-\lambda I & I & 0 \\ C & 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & B \\ 0 & D \end{bmatrix}. \tag{5.11}$$

Note that for any pair of matrices M and N we have $\operatorname{rank} MN = \operatorname{rank} N$ if and only if $\ker MN = \ker N$. By applying this to (5.11), we see that (5.9) is equivalent to

$$\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = 0 \implies \begin{bmatrix} I & 0 \\ 0 & B \\ 0 & D \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = 0.$$

It is straightforward to check that, in turn, this holds if and only if

$$\begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = 0 \implies \xi = 0. \tag{5.12}$$

Similarly, note that for all $A \in \mathcal{A}$

$$\begin{bmatrix} R-\lambda QP & QB \\ CP & D \end{bmatrix} = \begin{bmatrix} Q(A-\lambda I) & I & 0 \\ C & 0 & I \end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & QB \\ 0 & D \end{bmatrix}.$$

This makes (5.10) equivalent to

$$\begin{bmatrix} R-\lambda QP & QB \\ CP & D \end{bmatrix}\begin{bmatrix} \nu \\ \eta \end{bmatrix} = 0 \implies P\nu = 0. \tag{5.13}$$

From here on, we will prove the first statement of the theorem, noting any changes required for the second part.

[1] For a given subspace $\mathcal{L}$ and matrix M we denote by $M^{-1}\mathcal{L}$ the inverse image $\{x \mid Mx \in \mathcal{L}\}$.

($\Leftarrow$): Let $A \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ (resp. $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$). Assume that $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$ and (5.10) holds for λ. We will prove that (5.12) holds. For this, let ξ and η satisfy

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = 0.$$

Since $\xi \in C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$, we can write $\xi = P\nu$ for some ν. Now, by pre-multiplying with $\begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}$ we obtain that

$$\begin{bmatrix} R - \lambda QP & QB \\ CP & D \end{bmatrix} \begin{bmatrix} \nu \\ \eta \end{bmatrix} = 0.$$

We can now apply (5.13) and thus conclude that $\xi = P\nu = 0$. This proves that (5.12) holds.

($\Rightarrow$): Assume that (5.12) holds for all $A \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ (resp. $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$). We will first prove that $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$. Suppose this inclusion does not hold, i.e., there exists $\hat{x} \in C^{-1} \operatorname{im} D$ with $\hat{x} \notin \operatorname{im} P$. There exists a $\hat{u}$ such that $C\hat{x} + D\hat{u} = 0$. Take any $A \in \mathcal{A}$ and $\mu \in \mathbb{R}$ (resp. $\mu \in \mathbb{R}$ such that $|\mu| \geqslant 1$). Since $\hat{x} \notin \operatorname{im} P$ there exists a real $n \times n$ matrix A_0 such that $A_0 P = 0$ and $A_0 \hat{x} = -(A - \mu I)\hat{x} - B\hat{u}$. This implies that $QA_0P = 0$. Now define $\bar{A} := A + A_0$. Note that $\bar{A} \in \mathcal{A}$ and

$$\begin{bmatrix} \bar{A} - \mu I & B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{u} \end{bmatrix} = 0.$$

By (5.12), we see that $\hat{x} = 0$, which contradicts with $\hat{x} \notin \operatorname{im} P$. Therefore $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$.

We now move to proving (5.13). Let $\lambda \in \mathbb{C}$ (resp. $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$), and let ν and η satisfy

$$\begin{bmatrix} R - \lambda QP & QB \\ CP & D \end{bmatrix} \begin{bmatrix} \nu \\ \eta \end{bmatrix} = 0.$$

Denote $\xi = P\nu$, then we see that $C\xi + D\eta = 0$ and $(A - \lambda I)\xi + B\eta \in \ker Q$ for any $A \in \mathcal{A}$.

We will prove that (5.13) holds in three separate cases: First, we prove the statement for real λ. For complex λ we consider the cases where the real and complex parts of ξ are linearly dependent and where these are linearly independent.

First suppose that $\lambda \in \mathbb{R}$. Then, without loss of generality, ν and η are real, and as such ξ is real. Suppose that $\xi \neq 0$, and take any $A \in \mathcal{A}$. Let A_0 be any real $n \times n$ matrix such that $A_0\xi = -(A - \lambda I)\xi - B\eta$ and $QA_0P = 0$. Such a matrix exists as $-(A-\lambda I)\xi - B\eta \in \ker Q$ and $\xi \neq 0$. Now take $\bar{A} = A + A_0$. Then it is immediate that $\bar{A} \in \mathcal{A}$ and:

$$\begin{bmatrix} \bar{A}-\lambda I & B \\ C & D \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = 0.$$

As (5.12) holds for $\bar{A}$ by assumption, we see that $\xi = 0$, which leads to a contradiction. Therefore $\xi = 0$.

Now consider that case where $\lambda \notin \mathbb{R}$. Suppose that the real and complex parts of ξ are linearly dependent. Therefore, there exist real scalars $\alpha, \beta \in \mathbb{R}$ and a real vector r such that $\xi = (\alpha + i\beta)r$. Let $\hat{r} = (\alpha - i\beta)\xi = (\alpha^2 + \beta^2)r$. Let $A \in \mathcal{A}$. Then

$$\begin{bmatrix} Q(A-\lambda I) & QB \\ C & D \end{bmatrix} \begin{bmatrix} \hat{r} \\ (\alpha - i\beta)\eta \end{bmatrix} = 0.$$

Denote $\lambda = a + bi$, where $b \neq 0$, and $(\alpha - i\beta)\eta = \eta_1 + i\eta_2$. Then we see that: $Q(A - aI)\hat{r} + QB\eta_1 = -bQ\hat{r} + QB\eta_2 = 0$ and $C\hat{r} + D\eta_1 = D\eta_2 = 0$. Let $\mu \in \mathbb{R}$ (resp. $\mu \in \mathbb{R}$ such that $|\mu| \geqslant 1$). Note that

$$\begin{bmatrix} Q(A-\mu I) & QB \\ C & D \end{bmatrix} \begin{bmatrix} b\hat{r} \\ b\eta_1 + (\mu - a)\eta_2 \end{bmatrix} = 0.$$

As μ is real, we can now apply the previous part of the proof to note that $b\hat{r} = 0$, which holds only if $\xi = 0$.

Now suppose that $\xi = Pp + iPq$, where Pp and Pq are linearly independent. If we take any $A \in \mathcal{A}$, we know that $Q(A - \lambda I)\xi + QB\eta = 0$, and that we can denote $(A - \lambda I)\xi + B\eta = \zeta_1 + \zeta_2 i$, where $\zeta_1, \zeta_2 \in \ker Q$. Take A_0 any real map such that $A_0 Pp = -\zeta_1$, $A_0 Pq = -\zeta_2$ and $QA_0 P = 0$. Such a map exists as Pp and Pq are linearly independent. Now take $\bar{A} = A + A_0$, then $\bar{A} \in \mathcal{A}$ and clearly

$$\begin{bmatrix} \bar{A}-\lambda I & B \\ C & D \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = 0.$$

Using (5.12), this implies that $\xi = 0$. This is a contradiction with the fact that Pp and Pq are linearly independent. □

5.3 Informativity analysis

In this section we will apply Theorem 5.4 to obtain necessary and sufficient conditions for informativity of input-state-output data for the system properties listed in Section 5.1. For a given system (5.6) we will denote by $x(t, x_0, u)$ and $y(t, x_0, u)$ the state and output sequence corresponding to the initial state $x(0) = x_0$ and input sequence u.

5.3.1 Strong observability and strong detectability

We first briefly review the properties of strong observability and strong detectability.

Definition 5.5. The system (5.6) is called *strongly observable* if for each $x_0 \in \mathbb{R}^n$ and input sequence u the following holds: $y(t, x_0, u) = 0$ for all $t \in \mathbb{Z}_+$ implies that $x_0 = 0$. The system is called *strongly detectable* if for all $x_0 \in \mathbb{R}^n$ and every input sequence u the following holds: $y(t, x_0, u) = 0$ for all $t \in \mathbb{Z}_+$ implies that $\lim_{t\to\infty} x(t, x_0, u) = 0$.

For continuous-time systems, necessary and sufficient conditions for strong observability and strong detectability were formulated in [160]. It can be verified that also the discrete-time system (5.6) is strongly observable (strongly detectable) if and only if the pair $(C + DK, A + BK)$ is observable (detectable) for all K. It is also straightforward to verify the following.

Proposition 5.6. *The system* (5.6) *is strongly observable if and only if for all* $\lambda \in \mathbb{C}$

$$\operatorname{rank}\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = n + \operatorname{rank}\begin{bmatrix} B \\ D \end{bmatrix}. \tag{5.14}$$

The system (5.6) *is strongly detectable if and only if* (5.14) *holds for all* $\lambda \in \mathbb{C}$ *such that* $|\lambda| \geqslant 1$.

As in Section 5.2, we now consider the situation that only the matrices B, C and D are given, and that the matrix A can be any matrix from the affine set (5.5) with P, Q and R given matrices. By applying Theorem 5.4 we then get the following necessary and sufficient conditions for strong observability and strong detectability of *all* systems (5.6) with A ranging over the affine set $\mathcal{A}$.

Theorem 5.7 (Uniform rank condition)**.** *Let* (P, Q, R) *and* (B, C, D) *be given matrices. Then* (5.6) *is strongly observable for all* $A \in \mathcal{A}$ *if and only if* $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$ *and for all* $\lambda \in \mathbb{C}$ *we have*

$$\operatorname{rank}\begin{bmatrix} R - \lambda QP & QB \\ CP & D \end{bmatrix} = \operatorname{rank} P + \operatorname{rank}\begin{bmatrix} QB \\ D \end{bmatrix}. \tag{5.15}$$

Similarly, (5.6) *is strongly detectable for all* $A \in \mathcal{A}$ *if and only if* $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$ *and* (5.15) *holds for all* $\lambda \in \mathbb{C}$ *such that* $|\lambda| \geqslant 1$.

Proof. This follows immediately by combining Proposition 5.6 and Theorem 5.4. □

We will now apply the previous result to informativity of input-state-output data. Suppose the data are (U_-, X, Y_-). Recall Definition 5.4 of the affine set

$\mathcal{A}_\mathcal{D}$ of all $n \times n$ matrices A such that the data are consistent with the system (A, B, C, D, E, F). We want to obtain conditions under which the data are informative for strong observability and for strong detectability. To this end, let $[M \quad N]$ be any matrix such that

$$\ker[M \quad N] = \operatorname{im} \begin{bmatrix} E \\ F \end{bmatrix}. \tag{5.16}$$

Then we have $A \in \mathcal{A}_\mathcal{D}$ if and only if $R = MAX_-$ with

$$R := [M \quad N] \begin{bmatrix} X_+ - BU_- \\ Y_- - CX_- - DU_- \end{bmatrix}. \tag{5.17}$$

The following then immediately follows from Theorem 5.7.

Theorem 5.8. *The data (U_-, X, Y_-) are informative for strong observability if and only if $C^{-1} \operatorname{im} D \subseteq \operatorname{im} X_-$ and for all $\lambda \in \mathbb{C}$ we have*

$$\operatorname{rank} \begin{bmatrix} R - \lambda MX_- & MB \\ CX_- & D \end{bmatrix} = \operatorname{rank} X_- + \operatorname{rank} \begin{bmatrix} MB \\ D \end{bmatrix} \tag{5.18}$$

where R is given by (5.17).

The data are informative for strong detectability if and only if $C^{-1} \operatorname{im} D \subseteq \operatorname{im} X_-$ and (5.18) holds for all $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$.

In the case of independent process and measurement noise (see Remark 5.3), in which case $E = [E_1 \ 0]$ and $F = [0 \ F_2]$, we have $A \in \mathcal{A}_\mathcal{D}$ if and only if there exists a matrix W_{1-} such that $X_+ = AX_- + BU_- + E_1 W_{1-}$. Thus, $A \in \mathcal{A}_\mathcal{D}$ if and only if $R = MAX_-$ with

$$R := M[X_+ - BU_-] \tag{5.19}$$

and M such that $\ker M = \operatorname{im} E_1 = \operatorname{im} E$. In this case, the formulation of Theorem 5.8 holds verbatim with this M, and the new R given by (5.19).

Finally, for the special case $E = 0$ (the case with no process noise), we have $A \in \mathcal{A}_\mathcal{D}$ if and only if $R = AX_-$ with

$$R := X_+ - BU_-. \tag{5.20}$$

In that case, Theorem 5.8 holds verbatim with $M = I_n$ and R given by (5.20).

5.3.2 Observability and detectability

Next, we turn to characterizing informativity of the data for the properties of observability and detectability. Consider the system

$$x(t+1) = Ax(t), \ y(t) = Cx(t). \tag{5.21}$$

The Hautus test states that (5.21) is observable (detectable) if and only if

$$\operatorname{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n$$

for all $\lambda \in \mathbb{C}$ (for all $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$).

Now, take the situation that only C is known, that matrices P, Q and R are given, and that A can be any matrix from the affine set $\mathcal{A}$ given by (5.5). By applying Theorem 5.7 to the special case $B = 0$ and $D = 0$, we then obtain the following.

Corollary 5.9 (Uniform Hautus test)**.** *Let (P, Q, R) and C be given matrices. Then (5.21) is observable for all $A \in \mathcal{A}$ if and only if $\ker C \subseteq \operatorname{im} P$ and for any $\lambda \in \mathbb{C}$ we have*

$$\operatorname{rank} \begin{bmatrix} R - \lambda Q P \\ C P \end{bmatrix} = \operatorname{rank} P. \tag{5.22}$$

Similarly, (5.21) is detectable for all $A \in \mathcal{A}$ if and only if $\ker C \subseteq \operatorname{im} P$ and (5.22) holds for all $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$.

We now apply the previous result to the situation that input-state-output data on the system are available. As before, suppose that the data are (U_-, X, Y_-) and consider the affine set $\mathcal{A}_\mathcal{D}$ of all $n \times n$ matrices given by (5.4). The next result establishes conditions under which the data are informative for observability and for detectability.

Corollary 5.10. *Let (U_-, X, Y_-) be given input-state-output data. Let $[M \;\; N]$ be any matrix such that (5.16) holds. Let R be given by (5.17). The data are informative for observability if and only if $\ker C \subseteq \operatorname{im} X_-$ and for all $\lambda \in \mathbb{C}$ we have*

$$\operatorname{rank} \begin{bmatrix} R - \lambda M X_- \\ C X_- \end{bmatrix} = \operatorname{rank} X_-. \tag{5.23}$$

The data are informative for detectability if and only if $\ker C \subseteq \operatorname{im} X_-$ and (5.23) holds for all $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$.

Again, in the special case that the process noise and measurement noise are independent, Corollary 5.10 holds verbatim with M such that $\ker M = \operatorname{im} E$ and R given by (5.19). For the case that there is no process noise, in the rank test (5.23) we should take $M = I_n$ and R given by (5.20). This proves the claim made in Example 5.2.

Example 5.11. As an example, consider the system (5.1) with

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; D = 0, \; F = 0.$$

Suppose that the following data are given:

$$U_- = [1 \;\; 1], \;\; X = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \;\; Y_- = [0 \;\; 0]. \tag{5.24}$$

These data are indeed compatible with the true system, since (5.3) holds with $W_- = [0 \;\; 1]$. It is easily verified that

$$\mathcal{A}_\mathcal{D} = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$

We will check whether the data are informative for strong detectability. Take $M = [0 \;\; 1]$. Since $F = 0$ we have $R = M[X_+ - BU_-] = [0 \;\; 0]$, $MX_- = [0 \;\; 1]$, $CX_- = [0 \;\; 0]$, $MB = 1$. The condition $C^{-1} \operatorname{im} D \subseteq \operatorname{im} X_-$ is satisfied, so informativity for strong detectability holds if and only if

$$\operatorname{rank} \begin{bmatrix} 0 & -\lambda & 1 \\ 0 & 0 & 0 \end{bmatrix} = 2$$

for $|\lambda| \geqslant 1$, which is clearly not the case. We now check informativity for detectability. This requires $\ker C \subseteq \operatorname{im} X_-$ and

$$\operatorname{rank} \begin{bmatrix} 0 & -\lambda \\ 0 & 0 \end{bmatrix} = 1$$

for $|\lambda| \geqslant 1$. Both conditions indeed hold. On the other hand, the data are not informative for observability since the rank condition fails for $\lambda = 0$. If, in the example, we modify C and take $C = [0 \;\; 1]$, and accordingly $Y_- = [0 \;\; 1]$, then the data are still not informative for strong observability. In that case the rank condition does hold for all $\lambda \in \mathbb{C}$, but the condition $C^{-1} \operatorname{im} D \subseteq \operatorname{im} X_-$ is violated. ■

Remark 5.12. For the noiseless case, without proof we mention that if, apart from A_{true}, also the true matrix C (which we will call C_{true}) is unknown (but B and D are still known), then both for informativity for observability and detectability a necessary condition is that X_- has full row rank. As illustrated in Example 5.11, this is no longer the case if C_{true} is known. Since $X_+ = A_{\text{true}} X_- + BU_-$ and $Y_- = C_{\text{true}} X_- + DU_-$, this implies $A_{\text{true}} = (X_+ - BU_-)X_-^\sharp$ and $C_{\text{true}} = (Y_- - DU_-)X_-^\sharp$ for any right-inverse $X_-^\sharp$ of X_-. Hence, in that case the data are informative for observability (detectability) if and only if X_- has full row rank, and the pair $((Y_- - DU_-)X_-^\sharp, (X_+ - BU_-)X_-^\sharp)$ is observable (detectable). The unknown A_{true} and C_{true} are then uniquely determined by the data.

5.3.3 Strong controllability and strong stabilizability

For the system (5.6), the dual properties of strong observability and strong detectability are strong controllability and strong stabilizability. These properties can be defined in terms of trajectories of the system. Here, for brevity, we define (5.6) to be strongly controllable (strongly stabilizable) if the pair $(A+LC, B+LD)$ is controllable (stabilizable) for all L. From this it is immediate that (5.6) is strongly controllable (strongly stabilizable) if and only if the dual system $(A^\top, C^\top, B^\top, D^\top)$ is strongly observable (strongly detectable). As before, assume that B, C and D are given, but that A can be any matrix from the affine set $\mathcal{A} := \{A \in \mathbb{R}^{n\times n} \mid R = QAP\}$, where P, Q and R are given. Obviously, $A \in \mathcal{A}$ if and only if $A^\top$ satisfies $R^\top = P^\top A^\top Q^\top$. The above observations make the following a matter of course.

Corollary 5.13. *Let (P, Q, R) and (B, C, D) be given. Then (5.6) is strongly controllable for all $A \in \mathcal{A}$ if and only if $\ker Q \subseteq B \ker D$ and for all $\lambda \in \mathbb{C}$*

$$\operatorname{rank}\begin{bmatrix} R-\lambda QP & QB \\ CP & D \end{bmatrix} = \operatorname{rank} Q + \operatorname{rank}\begin{bmatrix} CP & D \end{bmatrix}. \tag{5.25}$$

Similarly, (5.6) is strongly stabilizable for all $A \in \mathcal{A}$ if and only if $\ker Q \subseteq B \ker D$ and (5.25) holds for all $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$.

Since a given pair (A, B) is controllable (stabilizable) if and only if the quadruple $(A, B, 0, 0)$ is strongly controllable (strongly stabilizable), the following also follows immediately.

Corollary 5.14 (Uniform Hautus test)**.** *Given (P, Q, R) and B, the pair (A, B) is controllable for all $A \in \mathcal{A}$ if and only if $\ker Q \subseteq \operatorname{im} B$ and for any $\lambda \in \mathbb{C}$*

$$\operatorname{rank}\begin{bmatrix} R-\lambda QP & QB \end{bmatrix} = \operatorname{rank} Q. \tag{5.26}$$

Furthermore (A, B) is stabilizable if and only if $\ker Q \subseteq \operatorname{im} B$ and (5.26) holds for all $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$.

By appying the above in the context of informativity, we immediately obtain the following.

Corollary 5.15. *Let $(M \quad N)$ be such that (5.16) holds. Given the data (U_-, X, Y_-), let R be given by (5.17). The data are informative for strong controllability if and only if $\ker M \subseteq \operatorname{im} B$ and for all $\lambda \in \mathbb{C}$ we have*

$$\operatorname{rank}\begin{bmatrix} R-\lambda MX_- & MB \\ CX_- & D \end{bmatrix} = \operatorname{rank} M + \operatorname{rank}[CX_- \quad D]. \tag{5.27}$$

The data are informative for strong stabilizability if and only if $\ker M \subseteq \operatorname{im} B$ and (5.27) holds for all $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$.

Corollary 5.16. *Let* $[M \quad N]$ *be such that* (5.16) *holds and let* R *be given by* (5.17)*. The data* (U_-, X, Y_-) *are informative for controllability if and only if* $\ker M \subseteq \operatorname{im} B$ *and for all* $\lambda \in \mathbb{C}$ *we have*

$$\operatorname{rank} \begin{bmatrix} R - \lambda M X_- & MB \end{bmatrix} = \operatorname{rank} M. \tag{5.28}$$

The data are informative for stabilizability if and only if $\ker M \subseteq \operatorname{im} B$ *and* (5.28) *holds for all* $\lambda \in \mathbb{C}$ *with* $|\lambda| \geqslant 1$*.*

As before, in the special case of independent process and measurement noise, Corollary 5.15 and 5.16 hold verbatim with M such that $\ker M = \operatorname{im} E$ and R given by (5.19). In this special case, the rank test for controllability and stabilizability can be simplified to $\operatorname{rank} M \begin{bmatrix} X_+ - \lambda X_- & B \end{bmatrix} = \operatorname{rank} M$ for all $\lambda \in \mathbb{C}$, and $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$, respectively.

If there is no process noise, in the rank tests (5.27) and (5.28) we should take $M = I_n$ and $R = X_+ - BU_-$. For this special case, the rank test for controllability and stabilizability can even be simplified to

$$\operatorname{rank} \begin{bmatrix} X_+ - \lambda X_- & B \end{bmatrix} = n \tag{5.29}$$

for all $\lambda \in \mathbb{C}$, and $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$, respectively. Note that this proves the claim made in Example 5.1

Remark 5.17. The rank test (5.29) can also be derived from Theorem 3.3. Indeed, that theorem states that all pairs (A, B) that satisfy the linear equation $X_+ = AX_- + BU_-$ are controllable if and only if $\operatorname{rank} \begin{bmatrix} X_+ - \lambda X_- \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$. This result can be applied to our set up, where we assume that only A is unknown and that B is given. Indeed, by defining 'new data' by

$$\tilde{X}_+ := \begin{bmatrix} X_+ & B \end{bmatrix}, \tilde{X}_- := \begin{bmatrix} X_- & 0 \end{bmatrix}, \tilde{U}_- := \begin{bmatrix} U_- & I_m \end{bmatrix} \tag{5.30}$$

we have that a matrix A satisfies $X_+ = AX_- + BU_-$ if and only if (A, B) satisfies $\tilde{X}_+ = A\tilde{X}_- + B\tilde{U}_-$. By applying Theorem 3.3. to the new data (5.30) we then get that (A, B) is controllable for all A satisfying $X_+ = AX_- + BU_-$ if and only if (5.29) holds.

Example 5.18. Again take as the true system the one specified in Example 5.11. Also, let the data be given by (5.24). Note that the condition $\ker M \subseteq \operatorname{im} B$ is violated, so the data are neither informative for strong controllability nor for strong stabilizability. They are also not informative for controllability or stabilizability. ■

5.4 A geometric approach to informativity

It is well known, see for example [160], that observability and strong observability also allow tests in terms of certain subspaces of the state space, more specifically, the unobservable subspace and weakly unobservable subspace. Properties of the weakly unobservable subspace also characterize left-invertibility of the system. In this section we will use these ideas to characterize informativity for strong observability, observability and left-invertibility.

Again consider the system (5.6). We call a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ output-nulling controlled invariant if

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subseteq \mathcal{V} \times \{0\} + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix} \tag{5.31}$$

(see [160]). Since any finite sum of such subspaces retains this property, there exists a unique largest output-nulling controlled invariant subspace, which will be denoted by $\mathcal{V}(A, B, C, D)$. This subspace is called the *weakly unobservable subspace* of the system (5.6). The system (5.6) is strongly observable if and only if $\mathcal{V}(A, B, C, D) = \{0\}$, see [160, pp. 159-160 and Theorem 7.16].

Now, again consider the situation that the matrices B, C and D are specified, but that A can be any matrix from the affine set $\mathcal{A}$ given by (5.5), where the matrices $P \in \mathbb{R}^{n\times r}$, $Q \in \mathbb{R}^{\ell\times n}$ and $R \in \mathbb{R}^{\ell\times r}$ are given. We consider the set of all subspaces $\mathcal{J} \subseteq \mathbb{R}^r$ that satisfy the following inclusion:

$$\begin{bmatrix} R \\ CP \end{bmatrix} \mathcal{J} \subseteq QP\mathcal{J} \times \{0\} + \operatorname{im} \begin{bmatrix} QB \\ D \end{bmatrix}. \tag{5.32}$$

It is easily verified that any finite sum of such subspaces $\mathcal{J}$ retains this property, and therefore there exists a largest subspace of $\mathbb{R}^r$ that satisfies the inclusion (5.32). We will denote this subspace by $\mathcal{J}^\star$.

Remark 5.19. It is straightforward to check that $\mathcal{J}^\star$ can be found from B, C, D, P, Q and R in at most r steps by letting $\mathcal{J}_0 = \mathbb{R}^r$, and iterating

$$\mathcal{J}_{t+1} = \begin{bmatrix} R \\ CP \end{bmatrix}^{-1} \left(QP\mathcal{J}_t \times \{0\} + \operatorname{im} \begin{bmatrix} QB \\ D \end{bmatrix} \right). \tag{5.33}$$

The following result will be instrumental in the remainder of this section.

Theorem 5.20. *Let (P, Q, R) and (B, C, D) be such that $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$. Then the following hold:*

(a) *For all $A \in \mathcal{A}$, we have $\mathcal{V}(A, B, C, D) \subseteq P\mathcal{J}^\star$.*

(b) *There exists $\bar{A} \in \mathcal{A}$ such that $\mathcal{V}(\bar{A}, B, C, D) = P\mathcal{J}^\star$.*

Proof. (a): Assume that $C^{-1}\operatorname{im} D \subseteq \operatorname{im} P$ holds. Let $A \in \mathcal{A}$ and let $\mathcal{V} \subseteq \mathbb{R}^n$ be an output nulling controlled invariant subspace. Note that $C\mathcal{V} \subseteq \operatorname{im} D$, and therefore there exists a subspace $\mathcal{J}$ such that $\mathcal{V} = P\mathcal{J}$. We now see that

$$\begin{bmatrix} R \\ CP \end{bmatrix} \mathcal{J} = \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} P\mathcal{J} \subseteq QP\mathcal{J} \times \{0\} + \operatorname{im} \begin{bmatrix} QB \\ D \end{bmatrix}.$$

Due to the definition of $\mathcal{J}^\star$, we then obtain $\mathcal{V}(A, B, C, D) \subseteq P\mathcal{J}^\star$.

(b): Let $\mathcal{J}$ satisfy (5.32). Then for any $A \in \mathcal{A}$ and $x \in P\mathcal{J}$ there exists $u \in \mathbb{R}^m$ such that:

$$Cx + Du = 0, \quad \text{and} \quad QAx + QBu \in QP\mathcal{J}.$$

This implies that

$$\operatorname{im}(Ax + Bu) \subseteq Q^{-1} \operatorname{im} Q(Ax + Bu) \subseteq Q^{-1}QP\mathcal{J} = P\mathcal{J} + \ker Q.$$

Now let $\{x_1, \ldots, x_k\}$ be a basis of the subspace $P\mathcal{J}$. By the previous discussion, for all $i = 1, \ldots, k$ there exists a $u_i \in \mathbb{R}^m$ such that $Cx_i + Du_i = 0$ and $Ax_i + Bu_i = y_i + z_i$, where $y_i \in P\mathcal{J}$ and $z_i \in \ker Q$. Let A_0 be any real $n \times n$ matrix such that $A_0 x_i = -z_i$ for $i = 1, \ldots, k$ and $QA_0P = 0$. Then, by defining $\bar{A} = A + A_0$ we see that $\bar{A} \in \mathcal{A}$. By definition $\bar{A}x_i + Bu_i = y_i \in P\mathcal{J}$, and therefore, by writing $\mathcal{V} = P\mathcal{J}$, we have:

$$\begin{bmatrix} \bar{A} \\ C \end{bmatrix} \mathcal{V} \subseteq \mathcal{V} \times \{0\} + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix}.$$

Hence $P\mathcal{J} \subseteq \mathcal{V}(\bar{A}, B, C, D)$, proving that $P\mathcal{J}^\star \subseteq \mathcal{V}(\bar{A}, B, C, D)$ and thus $P\mathcal{J}^\star = \mathcal{V}(\bar{A}, B, C, D)$ by (a). ☐

Using Theorem 5.20 we immediately obtain the following.

Theorem 5.21. *Let (B, C, D) and (P, Q, R) be given. Then the system (5.6) is strongly observable for all $A \in \mathcal{A}$ if and only if $C^{-1}\operatorname{im} D \subseteq \operatorname{im} P$ and $\mathcal{J}^\star \subseteq \ker P$.*

Proof. From Theorem 5.7 we see that $C^{-1}\operatorname{im} D \subseteq \operatorname{im} P$ is a necessary condition. The rest follows from Theorem 5.20. ☐

The procedure can be mimicked in order to characterize observability. For the system (5.21), the unobservable subspace $\mathcal{N}$ is the largest A-invariant subspace contained in $\ker C$, and (5.21) is observable if and only if $\mathcal{N} = \{0\}$. In the situation that only C and matrices (P, Q, R) are given, while A can be any matrix in the affine set $\mathcal{A}$, we should look at the largest subspace $\mathcal{L} \subseteq \mathbb{R}^r$ with the properties that

$$R\mathcal{L} \subseteq QP\mathcal{L} \text{ and } CP\mathcal{L} = \{0\}. \tag{5.34}$$

Denote this subspace by $\mathcal{L}^\star$ Then, we obtain the following corollary.

Corollary 5.22. *Given* (P, Q, R) *and* C*, then* (5.21) *is observable for all* $A \in \mathcal{A}$ *if and only if* $\ker C \subseteq \operatorname{im} P$ *and* $\mathcal{L}^\star \subseteq \ker P$*.*

The subspace $\mathcal{L}^\star$ is obtained in at most r steps by applying the iteration (5.33) with $B = 0$ and $D = 0$.

We now very briefly put the above in the context of informativity of input-state-output data. As before, let (U_-, X, Y_-) be the noisy data obtained from the system (5.1). Let $[M \quad N]$ be any matrix such that (5.16) holds. Then, by Theorem 5.20, these data are informative for strong observability of (5.6) if and only if $C^{-1} \operatorname{im} D \subseteq \operatorname{im} X_-$ and $\mathcal{J}^\star \subseteq \ker X_-$, where $\mathcal{J}^\star$ is the largest subspace satisfying (5.32) with R given by (5.17), $P = X_-$ and $Q = M$. Likewise, informativity for observability holds if and only if $\ker C \subseteq \operatorname{im} X_-$ and $\mathcal{L}^\star \subseteq \ker X_-$.

Obviously, the above can, again, be dualized to obtain alternative tests for informativity for controllability and strong controllability. We omit the details here. Instead, we will turn to informativity for the property of left-invertibility of the system (5.6) now. We briefly recall the definition.

Definition 5.23. The system (5.6) is called *left-invertible* if for each input sequence u the following holds: $y(t, 0, u) = 0$ for all $t \in \mathbb{Z}_+$ implies that $u(t) = 0$ for all $t \in \mathbb{Z}_+$.

The following characterization of left-invertibility was given in [160, Thm. 8.26].

Proposition 5.24. *The following are equivalent:*

(a) *The system* (5.6) *is left-invertible.*

(b) $\mathcal{V}(A, B, C, D) \cap B \ker D = \{0\}$ *and* $\begin{bmatrix} B \\ D \end{bmatrix}$ *has full column rank.*

The next result then, again, follows from Theorem 5.20.

Theorem 5.25. *Let* (P, Q, R) *and* (B, C, D) *be given. Assume that* $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$*. Then the system* (5.6) *is left-invertible for all* $A \in \mathcal{A}$ *if and only if* $P\mathcal{J}^\star \cap B \ker D = \{0\}$ *and* $\begin{bmatrix} B \\ D \end{bmatrix}$ *has full column rank.*

As before, this can immediately be applied in the context of informativity. We omit the details.

Remark 5.26. Note that Theorem 5.25 requires $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$, which, unfortunately, for left-invertibility for all $A \in \mathcal{A}$ is not a necessary condition. This can be seen, for example, by taking $D = I$. Then, regardless of our choice of (P, Q, R), B and C, we see that (5.6) is left-invertible for all $A \in \mathcal{A}$. However, in this case $C^{-1} \operatorname{im} D = \mathbb{R}^n$, so the condition $C^{-1} \operatorname{im} D \subseteq \operatorname{im} P$ is violated if P does not have full row rank.

To conclude this section, we note that Theorem 5.25 can be dualized in a straightforward way to obtain a characterization of right-invertibility for all $A \in \mathcal{A}$, and conditions for informativity of data for right-invertibility. Again, we omit the details.

To illustrate the the theory developed in this section we give the following example.

Example 5.27. Consider the system (5.1) with

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad D = 0, \quad F = 0.$$

Let data be given by

$$X = \begin{bmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \; U_- = \begin{bmatrix} 0 & 0 & 4 \end{bmatrix}, \; Y_- = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

Since there is only process noise, we should take M such that $\ker M = \operatorname{im} E$. Define

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$R = M[X_+ - BU_-] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easily verified that the set of all matrices consistent with the data is equal to

$$\mathcal{A}_{\mathcal{D}} = \left\{ \begin{bmatrix} a_{11} & 1 & 0 & 0 \\ a_{21} & 0 & 1 & 0 \\ a_{31} & 0 & 0 & 1 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \mid a_{ij} \in \mathbb{R} \right\}.$$

Note that $C^{-1} \operatorname{im} D \subseteq \operatorname{im} X_-$. In this case $\mathcal{J}^\star = \mathbb{R}^3$, and therefore the data are not informative for strong observability. On the other hand, $\mathcal{L}^\star = \{0\}$, proving that we do have informativity for observability.

If we modify our system by taking $B = e_i$, the ith standard basis vector in $\mathbb{R}^4$ $(i = 2, 3, 4)$, and adapt the data X accordingly, we get $\mathcal{J}^\star = \mathbb{R}^{4-i} \times \{0\}^{i-1}$. This means that $X_-\mathcal{J}^\star = \{0\}^i \times \mathbb{R}^{4-i}$. Thus, only for $i = 4$, the data are informative for strong observability. For $i = 2, 3, 4$ the data are informative for left-invertibility. ■

5.5 Notes and references

The results of this chapter are based on the publication [50]. They are a follow-up to those in Chapter 3. In Chapter 3, we have studied data-driven analysis of system properties using noise-free and noisy data, where the noise matrix was assumed to satisfy a bound dictated by a quadratic matrix inequality. In contrast, in this chapter, we have assumed that the noise is contained in a subspace. In addition, we have treated system properties that were not studied in this book before, like strong observability, strong detectability, and the dual properties of strong controllability and strong stabilizability.

These structural properties are relevant in a wide range of observer, filter and control design problems. For definitions and extensive treatments we refer to [69, 114, 117, 147, 153, 194], and [160] and the references therein.

Analysis of system properties based on data has been studied also in [94, 119, 182, 199], which deal with data-based controllability and observability analysis. Whereas in the present chapter general data sets are allowed, these references impose restrictions on the data. The paper [125] deals with the problem of determining stability properties of input-output systems using time series data.

Part II

Data-driven Control

6

Data-driven stabilization

This chapter deals with informativity for control. The control objective will be stabilization. The aim is to find controllers that stabilize some unknown linear system. In the situation that the data obtained from this unknown system do not determine the system uniquely, we need to find controllers that stabilize all systems that are consistent with the data. If the data enable us to find such controllers, they are called informative for stabilization. The subsequent design step is then to determine suitable controllers using only these informative data. In this chapter we will first study the design of static state feedback controllers based on noiseless input-state data. Next, we will consider the problem of designing dynamic output feedback controllers based on input-state-output data. We will also take a look at the situation that we only have input-output data. The final section of this chapter deals with the design of quadratically stabilizing static state feedback controllers based on noisy input-state data.

6.1 Stabilization by state feedback

Consider the model class $\mathcal{M}$ of all linear input-state systems of the form

$$x(t+1) = Ax(t) + Bu(t)$$

where x is the n-dimensional state and u is the m-dimensional input. Suppose that we collect input-state data on the time interval $[0, T]$, leading to data $\mathcal{D} := (U_-, X)$ as given by (2.1). The set $\Sigma_{\mathcal{D}}$ of all systems in $\mathcal{M}$ that are consistent with the data is then equal to $\Sigma_{(U_-,X)}$ defined by

$$\Sigma_{(U_-,X)} := \left\{ (A, B) \in \mathcal{M} \mid X_+ = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\}. \tag{6.1}$$

Again, by assumption we have $(A_{\text{true}}, B_{\text{true}}) \in \Sigma_{(U_-,X)}$, where $(A_{\text{true}}, B_{\text{true}})$ represents the true, unknown system.

In the context of stabilization by state feedback we take as the control objective $\mathcal{O}$: 'interconnection with a state feedback controller yields a stable closed loop system'. In line with Definition 2.4 we then have the following definition of informativity for stabilization by state feedback.

Definition 6.1. We say that the data (U_-, X) are *informative for stabilization by state feedback* if there exists a $K \in \mathbb{R}^{m\times n}$ such that $A + BK$ is stable for all $(A, B) \in \Sigma_{(U_-,X)}$.

In other words, the input-state data (U_-, X) are informative for stabilization by state feedback if there exists a single real $m \times n$ matrix K such that $A+BK$ is stable for all systems (A, B) that are consistent with the data.

At this point, one may wonder about the relation between informativity for stabilizability (as in Definition 3.2) and informativity for stabilization. It is clear that the data (U_-, X) are informative for stabilizability if (U_-, X) are informative for stabilization by state feedback. However, the reverse statement does not hold in general. This is due to the fact that all systems (A, B) in $\Sigma_{(U_-,X)}$ may be stabilizable, but there may not exist a *common* feedback gain K such that $A+BK$ is stable for all of these systems. This is further illustrated in the following example.

Example 6.2. Consider the scalar system

$$x(t + 1) = u(t)$$

where $x(t), u(t) \in \mathbb{R}$. Suppose that we collect data on the time interval $[0, 1]$, specifically, $x(0) = 0$, $u(0) = 1$ and $x(1) = 1$. This means that $U_- = \begin{bmatrix}1\end{bmatrix}$ and $X = \begin{bmatrix}0 & 1\end{bmatrix}$. It can be shown that $\Sigma_{(U_-,X)} = \{(a, 1) \mid a \in \mathbb{R}\}$. Clearly, all systems in $\Sigma_{(U_-,X)}$ are stabilizable. Nonetheless, the data are not informative for *stabilization*. This is because the systems $(-1, 1)$ and $(1, 1)$ in $\Sigma_{(U_-,X)}$ cannot be stabilized by the *same* controller of the form $u(t) = Kx(t)$. We conclude that informativity of the data for stabilizability does not imply informativity for stabilization by state feedback. ■

The notion of informativity for stabilization by state feedback is a specific example of informativity for control. As described in Problem 2.6 of the introduction, we will first find necessary and sufficient conditions for informativity for stabilization by state feedback. After this, we will design a corresponding controller, as described in Problem 2.7.

In order to be able to characterize informativity for stabilization, we first state a useful lemma. Recall that $(A, B) \in \mathcal{M}$ is consistent with the data (U_-, X) if and only if it satisfies the inhomogeneous equation appearing in (6.1). The solution set of the corresponding homogeneous equation is denoted by $\Sigma^{\text{hom}}_{(U_-,X)}$ and is equal to

$$\Sigma^{\text{hom}}_{(U_-,X)} := \left\{ (A_0, B_0) \mid 0 = \begin{bmatrix} A_0 & B_0 \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\}. \tag{6.2}$$

Lemma 6.3. *Suppose that the data (U_-, X) are informative for stabilization by state feedback and let K be a feedback gain such that $A+BK$ is stable for all $(A, B) \in \Sigma_{(U_-,X)}$. Then $A_0+B_0K = 0$ for all $(A_0, B_0) \in \Sigma^{hom}_{(U_-,X)}$. Equivalently,*

$$\operatorname{im}\begin{bmatrix} I \\ K \end{bmatrix} \subseteq \operatorname{im}\begin{bmatrix} X_- \\ U_- \end{bmatrix}.$$

Proof. We first prove that A_0+B_0K is *nilpotent* for all $(A_0, B_0) \in \Sigma^{\text{hom}}_{(U_-,X)}$. By hypothesis, $A+BK$ is stable for all $(A, B) \in \Sigma_{(U_-,X)}$. Let $(A, B) \in \Sigma_{(U_-,X)}$ and $(A_0, B_0) \in \Sigma^{\text{hom}}_{(U_-,X)}$ and define the matrices $F := A+BK$ and $F_0 := A_0+B_0K$. Then, the matrix $F + \alpha F_0$ is stable for all $\alpha \geqslant 0$. By dividing by α, it follows that, for all $\alpha \geqslant 1$, the spectral radius of the matrix

$$M_\alpha := \frac{1}{\alpha}F + F_0$$

is smaller than $1/\alpha$. From the continuity of the spectral radius by taking the limit as α tends to infinity, we see that $F_0 = A_0 + B_0K$ is nilpotent for all $(A_0, B_0) \in \Sigma^{\text{hom}}_{(U_-,X)}$. Note that we have

$$((A_0 + B_0K)^\top A_0, (A_0 + B_0K)^\top B_0) \in \Sigma^{\text{hom}}_{(U_-,X)}$$

whenever $(A_0, B_0) \in \Sigma^{\text{hom}}_{(U_-,X)}$. This means that $(A_0 + B_0K)^\top(A_0 + B_0K)$ is nilpotent. Since the only symmetric nilpotent matrix is the zero matrix, we see that $A_0 + B_0K = 0$ for all $(A_0, B_0) \in \Sigma^{\text{hom}}_{(U_-,X)}$. This is equivalent to

$$\ker\begin{bmatrix} X_-^\top & U_-^\top \end{bmatrix} \subseteq \ker\begin{bmatrix} I & K^\top \end{bmatrix}$$

which is equivalent to $\operatorname{im}\begin{bmatrix} I \\ K \end{bmatrix} \subseteq \operatorname{im}\begin{bmatrix} X_- \\ U_- \end{bmatrix}$. □

The previous lemma is instrumental in proving the following theorem that gives necessary and sufficient conditions for informativity for stabilization by state feedback.

Theorem 6.4. *The data (U_-, X) are informative for stabilization by state feedback if and only if the matrix X_- has full row rank and there exists a right inverse $X_-^\sharp$ of X_- such that $X_+X_-^\sharp$ is stable.*

Moreover, K is such that $A + BK$ is stable for all $(A, B) \in \Sigma_{(U_-,X)}$ if and only if $K = U_-X_-^\sharp$, where $X_-^\sharp$ satisfies the above properties. In that case, $A + BK = X_+X_-^\sharp$ for all $(A, B) \in \Sigma_{(U_-,X)}$.

Proof. To prove the 'if' part of the first statement, suppose that X_- has full row rank and there exists a right inverse $X_-^\sharp$ of X_- such that $X_+X_-^\sharp$ is stable. We define $K := U_-X_-^\sharp$. Next, we see that

$$X_+X_-^\sharp = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} X_-^\sharp = A + BK \tag{6.3}$$

for all $(A, B) \in \Sigma_{(U_-,X)}$. Therefore, $A + BK$ is stable for all $(A, B) \in \Sigma_{(U_-,X)}$. We conclude that the data (U_-, X) are informative for stabilization by state feedback, proving the 'if' part of the first statement. This also immediately proves 'if' part of the second statement as a byproduct.

Next, to prove the 'only if' part of the first statement, suppose that the data (U_-, X) are informative for stabilization by state feedback. Let K be such that $A + BK$ is stable for all $(A, B) \in \Sigma_{(U_-,X)}$. By Lemma 6.3 we know that

$$\operatorname{im} \begin{bmatrix} I \\ K \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} X_- \\ U_- \end{bmatrix}.$$

This implies that X_- has full row rank and there exists a right inverse $X_-^\sharp$ such that

$$\begin{bmatrix} I \\ K \end{bmatrix} = \begin{bmatrix} X_- \\ U_- \end{bmatrix} X_-^\sharp. \tag{6.4}$$

By (6.3), we obtain $A + BK = X_+X_-^\sharp$, which shows that $X_+X_-^\sharp$ is stable. This proves the 'only if' part of the first statement. Finally, by (6.4), the stabilizing feedback gain K is indeed of the form $K = U_-X_-^\sharp$, which also proves the 'only if' part of the second statement. □

Theorem 6.4 gives a characterization of all input-state data that are informative for stabilization by state feedback and provides a stabilizing controller. Nonetheless, the procedure to compute this controller might not be entirely satisfactory since it is not clear how to find a right inverse of X_- that makes $X_+X_-^\sharp$ stable. In general, X_- has many right inverses, and $X_+X_-^\sharp$ can be stable or unstable depending on the particular right inverse $X_-^\sharp$. To deal with this problem and to solve the design problem, we give a characterization of informativity for stabilization in terms of linear matrix inequalities. The feasibility of such LMIs can be verified using standard tools.

Theorem 6.5. *The data (U_-, X) are informative for stabilization by state feedback if and only if there exists a matrix $\Theta \in \mathbb{R}^{T\times n}$ satisfying*

$$X_-\Theta = (X_-\Theta)^\top \quad \textit{and} \quad \begin{bmatrix} X_-\Theta & X_+\Theta \\ \Theta^\top X_+^\top & X_-\Theta \end{bmatrix} > 0. \tag{6.5}$$

Moreover, K is such that $A+BK$ is stable for all $(A,B) \in \Sigma_{(U_-,X)}$ if and only if $K = U_-\Theta(X_-\Theta)^{-1}$ for some matrix Θ satisfying (6.5).

Example 6.6. Consider an unstable system $(A_{\text{true}}, B_{\text{true}})$, where A_{true} and B_{true} are given by

$$A_{\text{true}} = \begin{bmatrix} 1.5 & 0 \\ 1 & 0.5 \end{bmatrix}, \quad B_{\text{true}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We collect data from this system on a the time interval from $t=0$ to $t=2$, which results in the data matrices

$$X = \begin{bmatrix} 1 & 0.5 & -0.25 \\ 0 & 1 & 1 \end{bmatrix}, \quad U_- = \begin{bmatrix} -1 & -1 \end{bmatrix}.$$

Clearly, the matrix X_- is square and invertible, and it can be verified that

$$X_+X_-^{-1} = \begin{bmatrix} 0.5 & -0.5 \\ 1 & 0.5 \end{bmatrix}$$

is stable, since its eigenvalues are $\frac{1}{2}(1 \pm \sqrt{2}i)$. We conclude by Theorem 6.4 that the data (U_-, X) are informative for stabilization by state feedback. The same conclusion can be drawn from Theorem 6.5 since

$$\Theta = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

solves (6.5). Next, we can conclude from either Theorem 6.4 or Theorem 6.5 that the stabilizing feedback gain in this example is unique, and given by $K = U_-X_-^{-1} = \begin{bmatrix} -1 & -0.5 \end{bmatrix}$. Finally, it is worth noting that the data are not informative for system identification. In fact, $(A,B) \in \Sigma_{(U_-,X)}$ if and only if

$$A = \begin{bmatrix} 1.5 + a_1 & 0.5a_1 \\ 1 + a_2 & 0.5 + 0.5a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 + a_1 \\ a_2 \end{bmatrix}$$

for some $a_1, a_2 \in \mathbb{R}$. ■

Proof of Theorem 6.5. To prove the 'if' part of the first statement, suppose that there exists a Θ satisfying (6.5). In particular, this implies that $X_-\Theta > 0$. Therefore, X_- has full row rank. By taking a Schur complement and multiplying by -1, we obtain

$$X_+\Theta(X_-\Theta)^{-1}(X_-\Theta)(X_-\Theta)^{-1}\Theta^\top X_+^\top - X_-\Theta < 0.$$

Since $X_-\Theta$ is positive definite, this implies that $X_+\Theta(X_-\Theta)^{-1}$ is stable. In other words, there exists a right inverse $X_-^\sharp := \Theta(X_-\Theta)^{-1}$ of X_- such that

$X_+X_-^\sharp$ is stable. By Theorem 6.4, we conclude that (U_-, X) are informative for stabilization by state feedback, proving the 'if' part of the first statement. Using Theorem 6.4 once more, we see that $K := U_-\Theta(X_-\Theta)^{-1}$ stabilizes all systems in $\Sigma_{(U_-,X)}$, which in turn proves the 'if' part of the second statement.

Subsequently, to prove the 'only if' part of the first statement, suppose that the data (U_-, X) are informative for stabilization by state feedback. Let K be any feedback gain such that $A + BK$ is stable for all $(A, B) \in \Sigma_{(U_-,X)}$. By Theorem 6.4, X_- has full row rank and K is of the form $K = U_-X_-^\sharp$, where $X_-^\sharp$ is a right inverse of X_- such that $X_+X_-^\sharp$ is stable. The stability of $X_+X_-^\sharp$ implies the existence of a $P > 0$ such that

$$(X_+X_-^\sharp)P(X_+X_-^\sharp)^\top - P < 0.$$

Next, we define $\Theta := X_-^\sharp P$ and note that

$$X_+\Theta P^{-1}(X_+\Theta)^\top - P < 0.$$

Via the Schur complement we conclude that

$$\begin{bmatrix} P & X_+\Theta \\ \Theta^\top X_+^\top & P \end{bmatrix} > 0.$$

Since $X_-X_-^\sharp = I$, we see that $P = X_-\Theta$, which proves the 'only if' part of the first statement. Finally, by definition of Θ, we have $X_-^\sharp = \Theta P^{-1} = \Theta(X_-\Theta)^{-1}$. Recall that $K = U_-X_-^\sharp$, which shows that K is of the form $K = U_-\Theta(X_-\Theta)^{-1}$ for Θ satisfying (6.5). This proves the 'only if' part of the second statement and hence the proof is complete. □

In addition to the stabilizing controllers discussed in Theorems 6.4 and 6.5, we may also look for a controller of the form $u(t) = Kx(t)$ that stabilizes the system in *finite time*. Such a controller is called a *deadbeat controller* and is characterized by the property that $(A_{\text{true}} + B_{\text{true}}K)^t x_0 = 0$ for all $t \geqslant n$ and all $x_0 \in \mathbb{R}^n$. Thus, K is a deadbeat controller if and only if $A_{\text{true}} + B_{\text{true}}K$ is nilpotent. Then, analogous to the definition of informativity for stabilization by state feedback, we have the following definition of informativity for deadbeat control.

Definition 6.7. We say that the data (U_-, X) are *informative for deadbeat control* if there exists a feedback gain K such that $A + BK$ is nilpotent for all $(A, B) \in \Sigma_{(U_-,X)}$.

In other words, the data are informative for deadbeat control if there exists a real $m \times n$ matrix K such that $A + BK$ is nilpotent for all systems consistent with the data. Similarly to Theorem 6.4, we obtain the following necessary and sufficient conditions for informativity for deadbeat control.

Theorem 6.8. *The data (U_-, X) are informative for deadbeat control if and only if the matrix X_- has full row rank and there exists a right inverse $X_-^\sharp$ of X_- such that $X_+X_-^\sharp$ is nilpotent.*

Moreover, if this condition is satisfied then the feedback gain $K := U_-X_-^\sharp$ yields a deadbeat controller, that is, $A+BK$ is nilpotent for all $(A, B) \in \Sigma_{(U_-,X)}$.

Proof. The proof is similar to that of Theorem 6.4. For the 'only if' part, note that a square matrix is nilpotent if and only if it has only zero eigenvalues, which implies that a nilpotent matrix is stable. □

Remark 6.9. In order to check the existence of, and to compute a suitable right inverse $X_-^\sharp$ such that $X_+X_-^\sharp$ is nilpotent, we can proceed as follows. Since X_- has full row rank, we have $T \geqslant n$. We now distinguish two cases: $T = n$ and $T > n$. In the former case, X_- is nonsingular and we should just check whether $X_+X_-^{-1}$ is nilpotent.

In the latter case, there exist matrices $F \in \mathbb{R}^{T\times n}$ and $G \in \mathbb{R}^{T\times(T-n)}$ such that $\begin{bmatrix} F & G \end{bmatrix}$ is nonsingular and $X_- \begin{bmatrix} F & G \end{bmatrix} = \begin{bmatrix} I_n & 0_{n,(T-n)} \end{bmatrix}$. It is easily checked that $X_-^\sharp$ is a right inverse of X_- if and only if $X_-^\sharp = F + GH$ for some $H \in \mathbb{R}^{(T-n)\times n}$. Finding a right inverse $X_-^\sharp$ such that $X_+X_-^\sharp$ is nilpotent therefore amounts to finding H such that $X_+F + X_+GH$ is nilpotent, i.e. has only zero eigenvalues. Computation of such a matrix H amounts to a state feedback stabilization problem for the pair (X_+F, X_+G) with stability domain equal to $\{0\}$, or, equivalently, a state feedback deadbeat control problem for the pair (X_+F, X_+G).

6.2 Stabilization by dynamic output feedback

Whereas in the previous section we have considered stabilization by static state feedback using data obtained from input and state measurements, in the present section we will take also output measurements into account. In particular, we will consider the problem of stabilization by dynamic ouput feedback. We now first consider this problem based on input, state and output measurements. Subsequently, we turn our attention to the case of input-output data.

6.2.1 Stabilization using input, state and output data

Suppose that our model class $\mathcal{M}$ consists of all systems of the form

$$x(t+1) = Ax(t) + Bu(t) \tag{6.6a}$$
$$y(t) = Cx(t) + Du(t). \tag{6.6b}$$

Here, x is the n-dimensional state, u is the m-dimensional input and y is the p-dimensional output. The dimensions n, m and p are given, fixed, integers. The unknown, true system $\mathcal{S}$ belongs to the model class $\mathcal{M}$, and is given by

$$x(t+1) = A_{\text{true}}x(t) + B_{\text{true}}u(t) \tag{6.7a}$$
$$y(t) = C_{\text{true}}x(t) + D_{\text{true}}u(t). \tag{6.7b}$$

Suppose that we have collected input-state-output data on the time interval $[0, T]$. Let U_-, X, X_-, and X_+ be defined as in Section 2.2 and let Y_- be defined in a similar way as U_-. Our data are now given by $\mathcal{D} = (U_-, X, Y_-)$. Since these data are assumed to be generated by the true system $(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})$ we have

$$\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A_{\text{true}} & B_{\text{true}} \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}.$$

The set $\Sigma_{\mathcal{D}}$ of all systems that are consistent with these data is then given by:

$$\Sigma_{(U_-,X,Y_-)} := \left\{ (A, B, C, D) \mid \begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\}. \tag{6.8}$$

We want to design a stabilizing dynamic controller $\mathcal{K}$ of the form

$$z(t+1) = Kz(t) + Ly(t) \tag{6.9a}$$
$$u(t) = Mz(t). \tag{6.9b}$$

Here, the controller state z is q-dimensional, where the controller dimension q needs to be designed as well.

As design objective $\mathcal{O}$ we now take: 'interconnection with a dynamic controller yields a stable closed loop system'. For a given dynamic controller $\mathcal{K} = (K, L, M)$ of the form (6.9), the closed-loop system obtained from interconnecting the controller with any system $(A, B, C, D) \in \Sigma_{(U_-,X,Y_-)}$ is governed by the matrix

$$\begin{bmatrix} A & BM \\ LC & K + LDM \end{bmatrix}. \tag{6.10}$$

Informativity of input-state-output data for stabilization by dynamic output feedback requires the existence of a controller $\mathcal{K}$ that stabilizes all systems in $\mathcal{M}$ that are consistent with the data:

Definition 6.10. The data (U_-, X, Y_-) are called informative for stabilization by dynamic output feedback if there exists a controller $\mathcal{K} = (K, L, M)$ such that (6.10) is stable for all $(A, B, C, D) \in \Sigma_{(U_-,X,Y_-)}$.

As in the general case of informativity for control, we consider two consequent problems: first, to characterize informativity for stabilization by dynamic output feedback in terms of necessary and sufficient conditions on the data and next to design a controller based on these data.

To aid in solving these problems, we will first investigate the case where U_- does not have full row rank. In this case, we will show that the problem can be 'reduced' to the full row rank case. For this, we start with the observation that any $U_- \in \mathbb{R}^{m\times T}$ of row rank $k < m$ can be decomposed as $U_- = S\hat{U}_-$, where $S \in \mathbb{R}^{m\times k}$ has full column rank and $\hat{U}_- \in \mathbb{R}^{k\times T}$ has full row rank. We now have the following lemma:

Lemma 6.11. *Consider the data (U_-, X, Y_-) and the corresponding set of systems $\Sigma_{(U_-,X,Y_-)}$ consistent with these data given by (6.8). Let S be a matrix of full column rank such that $U_- = S\hat{U}_-$ with $\hat{U}_-$ a matrix of full row rank. Let $S^\sharp$ be a left inverse of S.*

Then the data (U_-, X, Y_-) are informative for stabilization by dynamic output feedback if and only if the data $(\hat{U}_-, X, Y_-)$ are informative for stabilization by dynamic output feedback.

In particular, if we let $\Sigma_{(\hat{U}_-,X,Y_-)}$ be the set of systems $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ consistent with the 'reduced' data set $(\hat{U}_-, X, Y_-)$[1], and if $\hat{K}$, $\hat{L}$ and $\hat{M}$ are real matrices of appropriate dimensions, then the following two statements hold:

(a) *If (K, L, M) stabilizes all systems in $\Sigma_{(U_-,X,Y_-)}$ then $(K, L, S^\sharp M)$ stabilizes all systems in $\Sigma_{(\hat{U}_-,X,Y_-)}$.*

(b) *If $(\hat{K}, \hat{L}, \hat{M})$ stabilizes all systems in $\Sigma_{(\hat{U}_-,X,Y_-)}$ then $(\hat{K}, \hat{L}, S\hat{M})$ stabilizes all systems in $\Sigma_{(U_-,X,Y_-)}$.*

Proof. First note that

$$\Sigma_{(\hat{U}_-,X,Y_-)} = \left\{ (\hat{A}, \hat{B}, \hat{C}, \hat{D}) \mid \begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} X_- \\ \hat{U}_- \end{bmatrix} \right\}.$$

We will start by proving the following two implications:

$$(A, B, C, D) \in \Sigma_{(U_-,X,Y_-)} \Longrightarrow (A, BS, C, DS) \in \Sigma_{(\hat{U}_-,X,Y_-)} \tag{6.11}$$

$$(\hat{A}, \hat{B}, \hat{C}, \hat{D}) \in \Sigma_{(\hat{U}_-,X,Y_-)} \Longrightarrow (\hat{A}, \hat{B}S^\sharp, \hat{C}, \hat{D}S^\sharp) \in \Sigma_{(U_-,X,Y_-)}. \tag{6.12}$$

To prove implication (6.11), assume that $(A, B, C, D) \in \Sigma_{(U_-,X,Y_-)}$. Then, by definition

$$\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}.$$

[1]Note that, here, $\hat{B}$ and $\hat{D}$ have dimensions $n \times k$ and $p \times k$, respectively.

From the definition of S, we have $U_- = S\hat{U}_-$. Substitution of this results in

$$\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ S\hat{U}_- \end{bmatrix} = \begin{bmatrix} A & BS \\ C & DS \end{bmatrix} \begin{bmatrix} X_- \\ \hat{U}_- \end{bmatrix}.$$

This implies that $(A, BS, C, DS) \in \Sigma_{(\hat{U}_-, X, Y_-)}$. The implication (6.12) can be proven similarly by substitution of $\hat{U}_- = S^\sharp U_-$.

To prove the lemma, suppose that the data (U_-, X, Y_-) are informative for stabilization by dynamic output feedback, and that K, L, and M are such that the matrix

$$\begin{bmatrix} A & BM \\ LC & K + LDM \end{bmatrix}$$

is stable for all $(A, B, C, D) \in \Sigma_{(U_-, X, Y_-)}$. If $(\hat{A}, \hat{B}, \hat{C}, \hat{D}) \in \Sigma_{(\hat{U}_-, X, Y_-)}$ then $(\hat{A}, \hat{B}S^\sharp, \hat{C}, \hat{D}S^\sharp) \in \Sigma_{(U_-, X, Y_-)}$ by (6.12). This means that the matrix

$$\begin{bmatrix} \hat{A} & \hat{B}S^\sharp M \\ L\hat{C} & K + L\hat{D}S^\sharp M \end{bmatrix}$$

is stable for all $(\hat{A}, \hat{B}, \hat{C}, \hat{D}) \in \Sigma_{(\hat{U}_-, X, Y_-)}$. Hence the data $(\hat{U}_-, X, Y_-)$ are informative for stabilization by dynamic measurement feedback and $(K, L, S^\sharp M)$ is a stabilizing controller for all systems in $\Sigma_{(\hat{U}_-, X, Y_-)}$. This proves the 'only if' part of the lemma and statement (a). The proofs of (b) and the 'if' part of the lemma are analogous and hence omitted. □

We will now solve the informativity and design problems under the condition that U_- has full row rank. Before embarking on this, we however first need to consider the issue of informativity for identification in the context of input-state-output data. We have already studied this property in the context of input-state data in Section 3.1. Here, we will extend these results to input-state-output data.

Recall that our model class $\mathcal{M}$ is now given by (6.6).

Definition 6.12. The input-state-output data (U_-, X, Y_-) are called informative for identification if $\Sigma_{(U_-, X, Y_-)}$ contains exactly one element.

In other words informativity for identification requires the set of systems consistent with the data to be a singleton. If this is the case, it only contains the unknown, true, system $(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})$.

The following theorem gives necessary and sufficient conditions for this to hold:

Theorem 6.13. *The data (U_-, X, Y_-) are informative for system identification if and only if*

$$\operatorname{rank} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = n + m. \tag{6.13}$$

Furthermore, if (6.13) *holds, there exists a matrix* $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ *such that*

$$\begin{bmatrix} X_- \\ U_- \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{6.14}$$

and, for any such right inverse, $A_{\text{true}} = X_+V_1$, $B_{\text{true}} = X_+V_2$, $C_{\text{true}} = Y_-V_1$ *and* $D_{\text{true}} = Y_-V_2$.

Proof. The proof is similar to that of Theorem 3.1. □

It turns out that, under the assumption that U_- has full row rank, a necessary condition for informativity for stabilization by dynamic output feedback is that the data (U_-, X, Y_-) are informative for identification. In fact, we have the following result.

Theorem 6.14. *Consider the data* (U_-, X, Y_-) *and assume that* U_- *has full row rank. Then* (U_-, X, Y_-) *are informative for stabilization by dynamic measurement feedback if and only if the following conditions are satisfied:*

(a) *We have*

$$\operatorname{rank} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = n + m$$

equivalently, there exists a matrix $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ *such that*

$$\begin{bmatrix} X_- \\ U_- \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

(b) *The pair* (X_+V_1, X_+V_2) *is stabilizable and* (Y_-V_1, X_+V_1) *is detectable.*

Moreover, if the above conditions are satisfied, a stabilizing controller (K, L, M) *can be constructed as follows:*

(i) *Select a matrix* M *such that* $X_+(V_1 + V_2M)$ *is stable.*

(ii) *Choose a matrix* L *such that* $(X_+ - LY_-)V_1$ *is stable.*

(iii) *Define* $K := (X_+ - LY_-)(V_1 + V_2M)$.

Remark 6.15. Under the condition that U_- has full row rank, Theorem 6.14 asserts that informativity for stabilization by dynamic output feedback holds if and only if the only system consistent with the data is the true system, and this true system is both stabilizable and detectable. The controller proposed in (i), (ii), (iii) is a so-called *observer-based* controller. The feedback gains M and L can be computed using standard methods, for example via pole placement or LMIs.

Proof of Theorem 6.14. To prove the 'if' part, suppose that conditions (a) and (b) are satisfied. This implies the existence of the matrices (K, L, M) as defined in items (i), (ii) and (iii). We will now show that these matrices indeed constitute a stabilizing controller. Note that by condition (a), $\Sigma_{(U_-,X,Y_-)} = \{(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})\}$ with

$$\begin{bmatrix} A_{\text{true}} & B_{\text{true}} \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix} = \begin{bmatrix} X_+V_1 & X_+V_2 \\ Y_-V_1 & Y_-V_2 \end{bmatrix}. \tag{6.15}$$

By definition of K, L and M, the matrices $A_{\text{true}} + B_{\text{true}}M$ and $A_{\text{true}} - LC_{\text{true}}$ are stable and $K = A_{\text{true}} + B_{\text{true}}M - LC_{\text{true}} - LD_{\text{true}}M$. This implies that (K, L, M) is a stabilizing controller for $(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})$ since the matrices

$$\begin{bmatrix} A_{\text{true}} & B_{\text{true}}M \\ LC_{\text{true}} & A_{\text{true}} + B_{\text{true}}M - LC_{\text{true}} \end{bmatrix} \text{ and } \begin{bmatrix} A_{\text{true}} + B_{\text{true}}M & B_{\text{true}}M \\ 0 & A_{\text{true}} - LC_{\text{true}} \end{bmatrix}$$

are similar and thus have the same eigenvalues. We conclude that (U_-, X, Y_-) are informative for stabilization by dynamic output feedback and that the recipe given by (i), (ii) and (iii) leads to a stabilizing controller (K, L, M).

It remains to prove the 'only if' part. To this end, suppose that the data (U_-, X, Y_-) are informative for stabilization by dynamic output feedback. Let (K, L, M) be such that

$$\begin{bmatrix} A & BM \\ LC & K + LDM \end{bmatrix}$$

is stable for all $(A, B, C, D) \in \Sigma_{(U_-,X,Y_-)}$. Let $\zeta \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$ be such that

$$\begin{bmatrix} \zeta^\top & \eta^\top \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = 0.$$

Note that $(A + \zeta\zeta^\top, B + \zeta\eta^\top, C, D) \in \Sigma_{(U_-,X,Y_-)}$ if $(A, B, C, D) \in \Sigma_{(U_-,X,Y_-)}$. Therefore, the matrix

$$\begin{bmatrix} A & BM \\ LC & K + LDM \end{bmatrix} + \alpha \begin{bmatrix} \zeta\zeta^\top & \zeta\eta^\top M \\ 0 & 0 \end{bmatrix}$$

is stable for all $\alpha \in \mathbb{R}$. We conclude that for $\alpha > 0$ the spectral radius of the matrix

$$W_\alpha := \frac{1}{\alpha} \begin{bmatrix} A & BM \\ LC & K + LDM \end{bmatrix} + \begin{bmatrix} \zeta\zeta^\top & \zeta\eta^\top M \\ 0 & 0 \end{bmatrix}$$

is smaller than $1/\alpha$. By taking the limit as $\alpha \to \infty$, we see that the spectral radius of $\zeta\zeta^\top$ must be zero due to the continuity of spectral radius. Therefore, ζ must be zero. Since U_- has full column rank, we can conclude that η must be zero too. This proves that condition (a) holds and therefore $\Sigma_{(U_-,X,Y_-)} = \{(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})\}$. Since the controller (K, L, M) stabilizes the system $(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})$, the pair $(A_{\text{true}}, B_{\text{true}})$ is stabilizable and $(C_{\text{true}}, A_{\text{true}})$ is detectable. By (6.15) we conclude that condition (b) is also satisfied. This proves the theorem. □

The following corollary follows from Lemma 6.11 and Theorem 6.14 and gives necessary and sufficient conditions for informativity for stabilization by dynamic output feedback. Note that we do not make any a priori assumptions on the rank of U_-.

Corollary 6.16. *Let S be any full column rank matrix such that $U_- = S\hat{U}_-$ with $\hat{U}_-$ full row rank k. The data (U_-, X, Y_-) are informative for stabilization by dynamic output feedback if and only if the following two conditions are satisfied:*

(a) *We have*

$$\operatorname{rank} \begin{bmatrix} X_- \\ \hat{U}_- \end{bmatrix} = n + k$$

equivalently, there exists a matrix $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ such that

$$\begin{bmatrix} X_- \\ \hat{U}_- \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

(b) *The pair (X_+V_1, X_+V_2) is stabilizable and (Y_-V_1, X_+V_1) is detectable.*

Moreover, if the above conditions are satisfied, a stabilizing controller (K, L, M) is constructed as follows:

(i) *Select a matrix $\hat{M}$ such that $X_+(V_1 + V_2\hat{M})$ is stable. Define $M := S\hat{M}$.*

(ii) *Choose a matrix L such that $(X_+ - LY_-)V_1$ is stable.*

(iii) *Define $K := (X_+ - LY_-)(V_1 + V_2\hat{M})$.*

Remark 6.17. In the previous corollary it is clear that the system matrices of the data-generating system are related to the data via

$$\begin{bmatrix} A_{\text{true}} & B_{\text{true}}S \\ C_{\text{true}} & D_{\text{true}}S \end{bmatrix} = \begin{bmatrix} X_+ \\ Y_- \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}.$$

Therefore the corollary shows that informativity for stabilization by dynamic output feedback requires that A_{true} and C_{true} can be identified uniquely from the data. However, this does not hold for B_{true} and D_{true} in general.

6.2.2 Stabilization using only input and output data

Again, consider the model class of all systems of the form (6.6) with fixed state space dimension n, input dimension m and output dimension p. The (unknown) true system is given by (6.7). When given input, state and output data (U_-, X, Y_-), any system (A, B, C, D) consistent with these data satisfies

$$\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}. \tag{6.16}$$

In this subsection, we will consider the situation where we have access to input and output measurements only. Moreover, we assume that the data are collected on the time interval $[0, T]$. This means that our data are of the form (U_-, Y_-), where U_- and Y_- are given by

$$U_- := U_{[0,T-1]} \tag{6.17a}$$
$$Y_- := Y_{[0,T-1]}. \tag{6.17b}$$

Again, we are interested in informativity of the data, this time given by (U_-, Y_-). To this end, we consider all systems in the model class $\mathcal{M}$ that admit the same input-output data[2] . This leads to the following set of systems that are consistent with the data:

$$\Sigma_{(U_-,Y_-)} := \left\{ (A, B, C, D) \mid \exists X \in \mathbb{R}^{n\times(T+1)} \text{ s.t. } (6.16) \text{ holds} \right\}.$$

As in the previous subsection, we wish to find a controller of the form (6.9) that stabilizes all these systems. In line with Definition 6.10, we have the following notion of informativity:

Definition 6.18. We say the input-output data (U_-, Y_-) are *informative for stabilization by dynamic measurement feedback* if there exist matrices K, L and M such that (6.10) is stable for all $(A, B, C, D) \in \Sigma_{(U_-,Y_-)}$.

In order to obtain conditions under which (U_-, Y_-) are informative for stabilization, it may be tempting to follow the same steps as in Section 6.2.1. There, we first proved that we can assume without loss of generality that U_- has full row rank. Subsequently, Theorem 6.14 and Corollary 6.16 characterize informativity for stabilization by dynamic measurement feedback based on input, state and output data. It turns out that we can perform the first of these two steps for input-output data as well. Indeed, in line with Lemma 6.11, we can state the following:

[2] We assume that the state space dimension n is known a priori.

Lemma 6.19. *Consider the data (U_-, Y_-) and the corresponding set $\Sigma_{(U_-,Y_-)}$. Let S be a matrix of full column rank such that $U_- = S\hat{U}_-$ with $\hat{U}_-$ a matrix of full row rank. Then the data (U_-, Y_-) are informative for stabilization by dynamic measurement feedback if and only if the data $(\hat{U}_-, Y_-)$ are informative for stabilization by dynamic measurement feedback.*

The proof of this lemma is analogous to that of Lemma 6.11 and is therefore omitted. Lemma 6.19 implies that without loss of generality we can consider the case where U_- has full row rank.

In contrast to the first step, the second step in Subsection 6.2.1 relies heavily on the affine structure of the considered set $\Sigma_{(U_-,X,Y_-)}$. However, the set $\Sigma_{(U_-,Y_-)}$ is not an affine set. This means that it is not straightforward to extend the results of Corollary 6.16 to the case of input-output measurements.

Nonetheless, under certain conditions on the input-output data it is possible to construct the corresponding state sequence X of (6.6) up to similarity transformation. As discussed in Subsection 1.2.2, state reconstruction is one of the main themes of the field of subspace identification. The construction of a state sequence would allow us to reduce the problem of stabilization using input-output data to that with input, state and output data. The following result gives sufficient conditions on the data (U_-, Y_-) for state construction.

To state the result, first recall from Subsection 1.2.2 the notation concerning Hankel matrices. Given input and output data $u_{[0,T-1]}$ and $y_{[0,T-1]}$, equivalently, the matrices $U_- = U_{[0,T-1]}$ and $Y_- = Y_{[0,T-1]}$, and k such that $2k \leqslant T$ we consider $H_{2k}(u_{[0,T-1]})$ and $H_{2k}(y_{[0,T-1]})$. Next, we partition our data into so-called 'past' and 'future' data as

$$H_{2k}(u_{[0,T-1]}) = \begin{bmatrix} U_p \\ U_f \end{bmatrix}, \quad H_{2k}(y_{[0,T-1]}) = \begin{bmatrix} Y_p \\ Y_f \end{bmatrix}$$

where U_p, U_f, Y_p and Y_f all have k block rows. Assume now that the true system (6.7) is observable, i.e. $(C_{\text{true}}, A_{\text{true}})$ is observable, and that $T \geqslant n - 1$. Then there exists a unique state trajectory $x_{[0,T]}$ of (6.7) corresponding to the finite input-output data $(u_{[0,T-1]}, y_{[0,T-1]})$. We now denote

$$\begin{aligned} X_p &= X_{[0,T-2k]} \\ X_f &= X_{[k,T-k]}. \end{aligned}$$

Lastly, recall that $\operatorname{rsp} M$ denotes the row space of the matrix M. The following theorem is a special case of Proposition 1.4.

Theorem 6.20. *Assume that the true system (6.7) is minimal, i.e. $(A_{\text{true}}, B_{\text{true}})$ is controllable and $(C_{\text{true}}, A_{\text{true}})$ is observable. Let the data (U_-, Y_-) be as in*

(6.17). *Assume that* T *and* k *are such that* $n \leqslant k \leqslant \frac{1}{2}T$. *If*

$$\operatorname{rank} \begin{bmatrix} H_{2k}(u_{[0,T-1]}) \\ H_{2k}(y_{[0,T-1]}) \end{bmatrix} = 2km + n \tag{6.18}$$

then

$$\operatorname{rsp} X_f = \operatorname{rsp} \begin{bmatrix} U_p \\ Y_p \end{bmatrix} \cap \operatorname{rsp} \begin{bmatrix} U_f \\ Y_f \end{bmatrix}$$

and this row space is of dimension n.

Clearly, under the conditions of this theorem, we can now recover the true state sequence X_f up to similarity transformation. That is, we can find $\bar{X} = QX_f$ for some unknown invertible matrix Q. This means that, under these conditions, we obtain an input-state-output trajectory given by the matrices

$$\bar{U}_- = U_{[k,T-k-1]} \tag{6.19a}$$

$$\bar{Y}_- = Y_{[k,T-k-1]} \tag{6.19b}$$

$$\bar{X} = QX_{[k,T-k]}. \tag{6.19c}$$

We can now state the following sufficient condition for informativity for stabilization with input-output data.

Corollary 6.21. *Assume that the true system* (6.7) *is minimal. Let the input-output data* (U_-, Y_-) *be as in* (6.17). *Assume that* T *and* k *are such that* $n \leqslant k \leqslant \frac{1}{2}T$. *Then the data* (U_-, Y_-) *are informative for stabilization by dynamic measurement feedback if the following two conditions are satisfied:*

(a) *The rank condition* (6.18) *holds.*

(b) *The data* $(\bar{U}_-, \bar{X}, \bar{Y}_-)$, *as defined in* (6.19), *are informative for stabilization by dynamic measurement feedback.*

Moreover, if these conditions are satisfied, a controller (K, L, M) *that stabilizes all systems in* $\Sigma_{(U_-,Y_-)}$ *can be found by applying Corollary 6.16 (i),(ii),(iii) to the data* $(\bar{U}_-, \bar{X}, \bar{Y}_-)$.

The conditions provided in Corollary 6.21 are sufficient, but not necessary for informativity for stabilization by dynamic measurement feedback. In addition, the data satisfying these conditions are also informative for identification, in the more general sense that $\Sigma_{(U_-,Y_-)}$ contains only the true system (6.7) and all systems similar to it.

6.3 Quadratic stabilization using noisy data

In the present section we will take a look at the situation that unknown process noise may enter the system. More specifically, we will study conditions under which the noisy data $\mathcal{D}$ as introduced in Section 3.4 are informative for quadratic stabilization. This will mean that all systems in the set $\Sigma_{\mathcal{D}}$ of systems consistent with the data can be stabilized by the same state feedback gain, with a *common Lyapunov function* for all closed loop systems. In particular then, this feedback gain will stabilize the unknown system. Conditions for the existence of such a feedback gain will be in terms of feasibility of certain linear matrix inequalities involving the data $\mathcal{D} = (U_-, X)$ and the (known) matrix Φ representing the quadratic inequality constraint on the matrix of noise samples. In addition, the controller gain will be computed in terms of solutions to these linear matrix inequalities.

Again we will consider the model class $\mathcal{M}$ of all noisy input-state systems with state dimension n and input dimension m of the form (3.17). We have input-state data (U_-, X) on the time interval $[0, T]$ and we assume that the possible matrices W_- of noise samples satisfy the quadratic inequality (3.18) for a given, known matrix $\Phi \in \mathbf{\Pi}_{n,T}$. As we have seen before, the set $\Sigma_{\mathcal{D}}$ of all systems consistent with the data is equal to the set of all systems (A, B) satisfying

$$X_+ = AX_- + BU_- + W_- \tag{6.20}$$

for some W_- satisfying (3.18), i.e.,

$$\Sigma_{\mathcal{D}} = \{(A, B) \mid (6.20) \text{ holds for some } W_- \text{ satisfying } (3.18)\}. \tag{6.21}$$

Definition 6.22. The data (U_-, X) are called *informative for quadratic stabilization* if there exist a feedback gain $K \in \mathbb{R}^{m\times n}$ and an $n \times n$ matrix $P > 0$ such that

$$P - (A + BK)P(A + BK)^\top > 0 \tag{6.22}$$

for all $(A, B) \in \Sigma_{\mathcal{D}}$.

We are interested in *quadratic stabilization* in the sense that we ask for a *common* Lyapunov matrix P for all $(A, B) \in \Sigma_{\mathcal{D}}$. Note that $P > 0$ satisfies (6.22) if and only $Q := P^{-1}$ satisfies $Q - (A + BK)^\top Q(A + BK) > 0$, which expresses that $V(x) = x^T Qx$ is a Lyapunov function for the system $x(t + 1) = (A + BK)x(t)$.

Definition 6.22 leads to two natural problems. First, we are interested in the question under which conditions the data are informative. We formalize this in the following problem.

Problem 6.23 (Informativity)**.** Find necessary and sufficient conditions under which the data (U_-, X) are informative for quadratic stabilization.

The second problem is the design issue: we are interested in procedures to come up with a feedback gain that stabilizes all systems in $\Sigma_{\mathcal{D}}$.

Problem 6.24 (Control design)**.** Given informative data (U_-, X), find a feedback gain K such that (6.22) is satisfied for all $(A, B) \in \Sigma_{\mathcal{D}}$.

Recall from Lemma 3.16 that $(A, B) \in \Sigma_{\mathcal{D}}$ if and only if

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} \geqslant 0. \tag{6.23}$$

Next, suppose that we fix a Lyapunov matrix $P > 0$ and a feedback gain K. The inequality (6.22) is equivalent to

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} P & 0 & 0 \\ 0 & -P & -PK^\top \\ 0 & -KP & -KPK^\top \end{bmatrix} \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} > 0 \tag{6.24}$$

which is also a quadratic matrix inequality in A and B. Therefore, finding conditions for quadratic stabilization as stated in Problem 6.23 amounts to finding conditions under which the quadratic matrix inequality (6.24) holds for all (A, B) satisfying the quadratic matrix inequality (6.23). As before, let N be defined by (3.30), and define

$$M := \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{12}^\top & M_{22} \end{array}\right] := \left[\begin{array}{c|cc} P & 0 & 0 \\ \hline 0 & -P & -PK^\top \\ 0 & -KP & -KPK^\top \end{array}\right]. \tag{6.25}$$

Then we need to find conditions on the data such that there exist $P > 0$ and K such that the inclusion

$$\mathcal{Z}_{n+m}(N) \subseteq \mathcal{Z}^+_{n+m}(M) \tag{6.26}$$

holds. In order to find such conditions, we will apply Corollary A.23. To do so, we need to verify its assumptions. In particular, we need to verify that $N_{22} \leqslant 0$, $\ker N_{22} \subseteq \ker N_{12}$, $N \,|\, N_{22} \geqslant 0$ and $M_{22} \leqslant 0$. The first three of these conditions indeed hold, as was already verified in Section 3.6. Also $M_{22} \leqslant 0$ since $P > 0$ and

$$M_{22} = -\begin{bmatrix} I \\ K \end{bmatrix} P \begin{bmatrix} I \\ K \end{bmatrix}^\top. \tag{6.27}$$

Corollary A.23 then asserts that (6.26) holds if and only if there exist scalars $\alpha \geqslant 0$ and $\beta > 0$ such that

$$M - \alpha N \geqslant \begin{bmatrix} \beta I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6.28}$$

From a design point of view, the matrices P and K that appear in M are not given. However, the idea is now to *compute* matrices P, K and scalars α and β such that (6.28) holds. In fact, by the above discussion, the data (U_-, X) are informative for quadratic stabilization *if and only if* there exist an $n \times n$ matrix $P > 0$, $K \in \mathbb{R}^{m \times n}$ and two scalars $\alpha \geqslant 0$ and $\beta > 0$ such that (6.28) holds. We note that (6.28) (in particular, M) is not linear in P and K. Nonetheless, by a rather standard change of variables and a Schur complement argument, we can transform (6.28) into a linear matrix inequality. Moreover, it turns out that the scalar α is necessarily positive. By a scaling argument then, it can be chosen to be equal to 1. We summarize our progress in the following theorem.

Theorem 6.25. *The data (U_-, X) are informative for quadratic stabilization if and only if there exist an $n \times n$ matrix $P > 0$, an $L \in \mathbb{R}^{m \times n}$ and a scalar $\beta > 0$ satisfying*

$$\begin{bmatrix} P - \beta I & 0 & 0 & 0 \\ 0 & -P & -L^\top & 0 \\ 0 & -L & 0 & L \\ 0 & 0 & L^\top & P \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \end{bmatrix}^\top \geqslant 0. \tag{6.29}$$

Moreover, if P and L satisfy (6.29) then $K := LP^{-1}$ is a stabilizing feedback gain for all $(A, B) \in \Sigma_{\mathcal{D}}$.

Proof. To prove the 'if' statement, suppose that there exist P, L and β satisfying (6.29). Define $K := LP^{-1}$. By computing the Schur complement of (6.29) with respect to its fourth diagonal block, we obtain (6.28) with $\alpha = 1$. As such, (6.26) holds. We conclude that the data (U_-, X) are informative for quadratic stabilization and $K = LP^{-1}$ is indeed a stabilizing controller for all $(A, B) \in \Sigma_{\mathcal{D}}$.

Conversely, to prove the 'only if' statement, suppose that the data (U_-, X) are informative for quadratic stabilization. This means that there exist $P > 0$ and K such that (6.26) holds. By Corollary A.23 there exist $\alpha \geqslant 0$ and $\beta > 0$ satisfying (6.28). Then, by defining $L := KP$ and using a Schur complement argument, we conclude that the LMI

$$\begin{bmatrix} P - \beta I & 0 & 0 & 0 \\ 0 & -P & -L^\top & 0 \\ 0 & -L & 0 & L \\ 0 & 0 & L^\top & P \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \end{bmatrix}^\top \geqslant 0$$

is feasible. Zooming in on the $(2,2)$ block, this yields $-P - \alpha X_- \Phi_{22} X_-^\top \geqslant 0$. Since $P > 0$ this implies $\alpha > 0$. As a consequence, by scaling P, L and β by $\frac{1}{\alpha}$ we may assume that $\alpha = 1$, so the LMI (6.29) is feasible. □

Theorem 6.25 provides a powerful necessary *and* sufficient condition under which quadratically stabilizing controllers can be obtained from noisy data. The theorem leads to an effective design procedure for obtaining stabilizing controllers directly from data. Indeed, the approach entails solving the linear matrix inequality (6.29) for P, L and β and computing a controller as $K = LP^{-1}$. Below, we discuss some of the features of our control design procedure.

(i) First of all, we stress that the procedure is *non-conservative* since Theorem 6.25 provides a necessary and sufficient condition for obtaining quadratically stabilizing controllers from data.

(ii) The variables P, L and β are *independent* of the time horizon T of the experiment. In fact, note that $P \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{m\times n}$ and $\beta \in \mathbb{R}$. Also, the LMI (6.29) is of dimension $(3n+m)\times(3n+m)$ and thus independent of T. This T-independent design method can play a crucial role in control design from larger data sets. We note that collections of big data sets are often unavoidable, for example because the signal-to-noise ratio is small, or because the data-generating system is large-scale.

We note that, in an analogous way as Theorem 3.18, under the extra assumptions $\Phi_{22} < 0$ and

$$\operatorname{rank}\begin{bmatrix} X_- \\ U_- \end{bmatrix} = n + m \tag{6.30}$$

it is possible to prove a variant of Theorem 6.25 in which the non-strict inequality is replaced by a strict inequality, and the term $-\beta I$ is removed. This can be done by invoking Theorem A.20, which is possible since the conditions $\Phi_{22} < 0$ and (6.30) yield $N_{22} < 0$. Thus we obtain the following theorem.

Theorem 6.26. *Assume that $\Phi_{22} < 0$ and the rank condition (6.30) holds. Then the data (U_-, X) are informative for quadratic stabilization if and only if there exist an $n \times n$ matrix $P > 0$ and an $L \in \mathbb{R}^{m\times n}$ satisfying*

$$\begin{bmatrix} P & 0 & 0 & 0 \\ 0 & -P & -L^\top & 0 \\ 0 & -L & 0 & L \\ 0 & 0 & L^\top & P \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \end{bmatrix}^\top > 0. \tag{6.31}$$

Moreover, if P and L satisfy (6.31) then $K := LP^{-1}$ is a stabilizing feedback gain for all $(A, B) \in \Sigma_{\mathcal{D}}$.

Following up on this theorem, we will show that under the assumption that $\Phi_{22} < 0$, if the data (U_-, X) are informative for quadratic stabilization and if K stabilizes all systems in $\Sigma_{\mathcal{D}}$ with a common Lyapunov matrix $P > 0$, then, in fact, X_- must have full row rank, and K must be of the form $K = U_- X_-^\sharp$ for some right inverse $X_-^\sharp$ of X_-. Thus, the following theorem extends Lemma 6.3 to the noisy case.

Theorem 6.27. *Assume $\Phi_{22} < 0$. Let the data (U_-, X_-) be informative for quadratic stabilization and suppose that $P > 0$ and K are such that (6.26) holds. Then*

$$\operatorname{im} \begin{bmatrix} I \\ K \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} X_- \\ U_- \end{bmatrix}. \tag{6.32}$$

Consequently, X_- has full row rank n and there exists a right-inverse $X_-^\sharp$ of X_- such that $K = U_- X_-^\sharp$.

Proof. We will first prove that $\ker N_{22} \subseteq \ker M_{22}$. Let $Z := \begin{bmatrix} A^\top \\ B^\top \end{bmatrix} \in \mathcal{Z}_{n+m}(N)$. Let $\hat{Z} \in \mathbb{R}^{(n+m)\times n}$ be such that $N_{22}\hat{Z} = 0$. Since $\ker N_{22} \subseteq \ker N_{12}$ we have $Z + \gamma \hat{Z} \in \mathcal{Z}_{n+m}(N)$ for any $\gamma \in \mathbb{R}$. Thus, we obtain

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} + \gamma(M_{12} + Z^\top M_{22})\hat{Z} + \gamma\left((M_{12} + Z^\top M_{22})\hat{Z}\right)^\top + \gamma^2 \hat{Z}^\top M_{22}\hat{Z} > 0. \tag{6.33}$$

We will argue that $M_{22}\hat{Z} = 0$. Indeed, recall that $M_{22} \leqslant 0$. Thus, if we assume that $M_{22}\hat{Z} \neq 0$ then there exists a sufficiently large γ such that (6.33) is violated. Next, since $P > 0$, by (6.27) we have $\ker M_{22} = \ker \begin{bmatrix} I & K^\top \end{bmatrix}$. Also, since $\Phi_{22} < 0$ and

$$N_{22} = \begin{bmatrix} X_- \\ U_- \end{bmatrix} \Phi_{22} \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top$$

we have

$$\ker N_{22} = \ker \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top.$$

Thus $\ker \begin{bmatrix} X_-^\top & U_-^\top \end{bmatrix} \subseteq \ker \begin{bmatrix} I & K^\top \end{bmatrix}$, equivalently

$$\operatorname{im} \begin{bmatrix} I \\ K \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} X_- \\ U_- \end{bmatrix}.$$

Therefore, any controller K that stabilizes all systems in $\Sigma_{\mathcal{D}}$ with a common Lyapunov matrix P is necessarily of the form $K = U_- X_-^\sharp$ for some right inverse $X_-^\sharp$ of X_-. ☐

6.3.1 Reducing computational complexity

The computational complexity of determining feasibility of an LMI depends on the size of the LMI and the number of unknowns. The LMI (6.29), together with the constraints $P > 0$ and $\beta > 0$, has size $4n + m + 1$ and contains $\frac{n(n+1)}{2} + nm + 1$ unknowns. Using Theorem A.7, we can separate the computation of the Lyapunov matrix P and the controller K. Below it will be shown that this leads to another LMI with size $4n+1$ and $\frac{n(n+1)}{2} + 1$ unknowns. This result thus has a significant computational advantage over Theorem 6.25.

Theorem 6.28. *Let $\Theta := \Phi_{12} + X_+\Phi_{22}$. The following statements hold.*

(a) *The data (U_-, X) are informative for quadratic stabilization if and only if there exist an $n \times n$ matrix $P > 0$ and a scalar $\beta > 0$ satisfying*

$$P - \beta I - \begin{bmatrix} I & X_+ \end{bmatrix} \Phi \begin{bmatrix} I \\ X_+^\top \end{bmatrix} + \Theta \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top \left(\begin{bmatrix} X_- \\ U_- \end{bmatrix} \Phi_{22} \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top \right)^\dagger \begin{bmatrix} X_- \\ U_- \end{bmatrix} \Theta^\top \geqslant 0 \tag{6.34a}$$

$$\begin{bmatrix} P - \beta I & 0 \\ 0 & -P \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix}^\top \geqslant 0. \tag{6.34b}$$

Moreover, if $P > 0$ and $\beta > 0$ satisfy (6.34a) and (6.34b) then

$$K = \left(U_-(\Phi_{22} + \Theta^\top \Gamma^\dagger \Theta) X_-^\top\right) \left(X_-(\Phi_{22} + \Theta^\top \Gamma^\dagger \Theta) X_-^\top\right)^\dagger \tag{6.35}$$

is a stabilizing gain for all $(A, B) \in \Sigma_\mathcal{D}$, where $\Gamma = P - \beta I - \begin{bmatrix} I & X_+ \end{bmatrix} \Phi \begin{bmatrix} I \\ X_+^\top \end{bmatrix}$.

(b) *Assume, in addition, that $\Phi_{22} < 0$ and $\operatorname{rank} \begin{bmatrix} X_-^\top & U_-^\top \end{bmatrix}^\top = n + m$. Then the data (U_-, X) are informative for quadratic stabilization if and only if there exists an $n \times n$ matrix $P > 0$ satisfying*

$$P - \begin{bmatrix} I & X_+ \end{bmatrix} \Phi \begin{bmatrix} I \\ X_+^\top \end{bmatrix} + \Theta \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top \left(\begin{bmatrix} X_- \\ U_- \end{bmatrix} \Phi_{22} \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top \right)^{-1} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \Theta^\top > 0 \tag{6.36a}$$

$$\begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix}^\top > 0. \tag{6.36b}$$

Moreover, if $P > 0$ satisfies (6.36a) and (6.36b) then K in (6.35) is a stabilizing feedback gain for all systems $(A, B) \in \Sigma_\mathcal{D}$, where Γ is defined as $\Gamma := P - \begin{bmatrix} I & X_+ \end{bmatrix} \Phi \begin{bmatrix} I \\ X_+^\top \end{bmatrix}$.

Proof. We first prove (a). Let $P > 0$ be an $n \times n$ matrix and $\beta > 0$ be a real number. According to Theorem 6.25, it is enough to show that there exists $L \in \mathbb{R}^{m\times n}$ satisfying (6.29) if and only if (6.34a) and (6.34b) are satisfied. By taking the Schur complement of the left hand side in (6.29) with respect to P, one can see that there exists $L \in \mathbb{R}^{m\times n}$ satisfying (6.29) if and only if there exists $L \in \mathbb{R}^{m\times n}$ satisfying

$$\begin{bmatrix} P-\beta I & 0 & 0 \\ 0 & -P & -L^\top \\ 0 & -L & -LP^{-1}L^\top \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top \geqslant 0. \tag{6.37}$$

By direct inspection, one can verify that (6.37) is equivalent to the following QMI:

$$\begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_m \\ 0 & 0 & P^{-1}L^\top \end{bmatrix}^\top \underbrace{\begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}}_{:=\Psi} \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_m \\ 0 & 0 & P^{-1}L^\top \end{bmatrix} \geqslant 0 \tag{6.38}$$

where

$$\Psi_{11} = \begin{bmatrix} P-\beta I & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top, \quad \Psi_{12} = \begin{bmatrix} 0 \\ -P \\ 0 \end{bmatrix},$$

and $\Psi_{22} = -P$. Observe that (6.38) is equivalent to

$$\begin{bmatrix} 0 & P^{-1}L^\top \end{bmatrix} \in \mathcal{Z}_n(\Psi). \tag{6.39}$$

Now, by defining $W = \begin{bmatrix} I_{2n} & 0 \end{bmatrix}^\top$ and $Y = 0$ it follows from Corollary A.9 that there exists an L such that (6.39) holds if and only if

$$\Psi \in \mathbf{\Pi}_{2n+m,n} \quad \text{and} \quad 0 \in \mathcal{Z}_n(\Psi_W) \tag{6.40}$$

where Ψ_W is defined in (A.19). Next, we observe that $\Psi \in \mathbf{\Pi}_{2n+m,n}$ if and only if $\Psi \,|\, \Psi_{22} \geqslant 0$ since $\Psi_{22} = -P < 0$. Note that

$$\Psi \,|\, \Psi_{22} = \begin{bmatrix} P-\beta I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top.$$

By a Schur complement argument, we see that $\Psi \,|\, \Psi_{22} \geqslant 0$ if and only if (6.34a) holds. Now, observe that $0 \in \mathcal{Z}_n(\Psi_W)$ if and only if $W^\top \Psi_{11} W \geqslant 0$. Note that

$$W^\top \Psi_{11} W = \begin{bmatrix} P-\beta I & 0 \\ 0 & -P \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix}^\top. \tag{6.41}$$

Therefore, $0 \in \mathcal{Z}_n(\Psi_W)$ if and only if (6.34b) holds. Consequently, the data are informative for quadratic stabilization if and only if there exists an $n \times n$ matrix $P > 0$ and a scalar $\beta > 0$ satisfying (6.34a) and (6.34b).

For the construction of the controller, assume that $P > 0$ and $\beta > 0$ satisfy (6.34a) and (6.34b). By (6.41), this implies that $W^\top \Psi_{11} W \geqslant 0$. This is equivalent to $-W^\top \Psi_{12}\Psi_{22}^{-1}\Psi_{21} W \leqslant W^\top (\Psi \,|\, \Psi_{22}) W$.

Using Lemma A.1 with $A = -(-\Psi_{22})^{-\frac{1}{2}}\Psi_{21}W$ and $B = (\Psi \,|\, \Psi_{22})^{\frac{1}{2}} W$ there exists a matrix $S \in \mathbb{R}^{n\times(2n+m)}$ satisfying $S^\top S \leqslant I$ and

$$-(-\Psi_{22})^{-\frac{1}{2}}\Psi_{21}W = S(\Psi \,|\, \Psi_{22})^{\frac{1}{2}} W. \tag{6.42}$$

In fact, the matrix $S := -(-\Psi_{22})^{-\frac{1}{2}}\Psi_{21}W \left((\Psi \,|\, \Psi_{22})^{\frac{1}{2}} W\right)^\dagger$ works. Since $\Psi_{22} < 0$ and $S^\top S \leqslant I$, Theorem A.6 yields $Z := -\Psi_{22}^{-1}\Psi_{21} + (-\Psi_{22})^{-\frac{1}{2}} S (\Psi \,|\, \Psi_{22})^{\frac{1}{2}} \in \mathcal{Z}_n(\Psi)$. It follows from (6.42) that $ZW = 0$. Now define $K := \begin{bmatrix} 0 & I_m \end{bmatrix} Z^\top$. Then, by (6.39) and Theorem 6.25, K is a stabilizing feedback for all $(A, B) \in \Sigma_{\mathcal{D}}$. It remains to be shown that K is equal to (6.35).

First, observe that

$$Z \begin{bmatrix} 0 \\ I_m \end{bmatrix} = \Psi_{22}^{-1}\Psi_{21}W \left((\Psi \,|\, \Psi_{22})^{\frac{1}{2}} W\right)^\dagger (\Psi \,|\, \Psi_{22})^{\frac{1}{2}} \begin{bmatrix} 0 \\ I_m \end{bmatrix}$$

because $\Psi_{21} \begin{bmatrix} 0 \\ I_m \end{bmatrix} = 0$. It can be shown that

$$\left((\Psi \,|\, \Psi_{22})^{\frac{1}{2}} W\right)^\dagger = (\Psi_W \,|\, \Psi_{22})^\dagger W^\top (\Psi \,|\, \Psi_{22})^{\frac{1}{2}}.$$

Then, using the fact that $\Psi_{22}^{-1}\Psi_{21}W = \begin{bmatrix} 0 & I_n \end{bmatrix}$, this yields

$$Z \begin{bmatrix} 0 \\ I_m \end{bmatrix} = \begin{bmatrix} 0 & I_n \end{bmatrix} (\Psi_W \,|\, \Psi_{22})^\dagger W^\top (\Psi \,|\, \Psi_{22}) \begin{bmatrix} 0 \\ I_m \end{bmatrix}$$

which implies

$$K = \begin{bmatrix} 0 & I_m \end{bmatrix} (\Psi \,|\, \Psi_{22}) W (\Psi_W \,|\, \Psi_{22})^\dagger \begin{bmatrix} 0 \\ I_n \end{bmatrix}. \tag{6.43}$$

Observe that

$$\begin{bmatrix} 0 & I_m \end{bmatrix} (\Psi \,|\, \Psi_{22}) W = U_- \begin{bmatrix} \Theta^\top & -\Phi_{22} X_-^\top \end{bmatrix}. \tag{6.44}$$

Moreover, note that $\Psi_W \mid \Psi_{22} = W^\top (\Psi \mid \Psi_{22}) W = \begin{bmatrix} \Gamma & \Theta X_-^\top \\ X_-\Theta^\top & \Omega \end{bmatrix}$, where $\Omega = -X_-\Phi_{22}X_-^\top$. It follows from [159, Thm. 2.10] that

$$(\Psi_W \,|\, \Psi_{22})^\dagger \begin{bmatrix} 0 \\ I_n \end{bmatrix} = \begin{bmatrix} \Gamma^\dagger \Theta X_-^\top \\ -I_n \end{bmatrix} \left(X_-(\Phi_{22} + \Theta^\top \Gamma^\dagger \Theta) X_-^\top\right)^\dagger. \tag{6.45}$$

By substituting (6.44) and (6.45) into (6.43), we see that K is equal to (6.35).

The proof of (b) can be established by following the same steps as the proof of (a), but replaces the term $P - \beta I$ by P and non-strict inequalities by strict ones. Note that in this case, the proof of the if and only if statement relies on Corollary A.10 rather than Corollary A.9 and on Theorem 6.26 instead of Theorem 6.25. Also the construction of the controller builds on the results for strict inequalities in Lemma A.1(b) and Theorem A.6(b), rather than their non-strict counterparts. This proves the theorem. □

6.3.2 Illustrative example: bounds on the noise samples

In this subsection we illustrate the theory on stabilization using noisy data as developed in this section before. Consider an unstable system of the form (3.17) with the true but unknown system matrices A_{true} and B_{true} given by

$$A_{\text{true}} = \begin{bmatrix} 0.850 & -0.038 & -0.380 \\ 0.735 & 0.815 & 1.594 \\ -0.664 & 0.697 & -0.064 \end{bmatrix}, \quad B_{\text{true}} = \begin{bmatrix} 1.431 & 0.705 \\ 1.620 & -1.129 \\ 0.913 & 0.369 \end{bmatrix}.$$

In this example, we assume that the noise samples $w(t)$ are bounded in norm as $\|w(t)\|_2^2 \leqslant \varepsilon$ for all t. As explained in Section 3.4, we can capture this prior knowledge using the noise model (3.18) with $\Phi_{11} = T\varepsilon I$, $\Phi_{12} = 0$ and $\Phi_{22} = -I$, where T is the time horizon used for data sampling.

In this example, we pick a time horizon of $T = 20$ and draw the entries of the inputs and initial state randomly from a Gaussian distribution with zero mean and unit variance. The noise samples are drawn uniformly at random from the ball $\{w \in \mathbb{R}^3 \mid \|w\|_2^2 \leqslant \varepsilon\}$. We aim at constructing stabilizing controllers from the input-state data for various values of ε. In particular, we investigate six different noise levels: $\varepsilon \in \{0.5, 1, 1.5, 2, 2.2, 2.4\}$. For each noise level, we generate 100 data sets using the method described above.

For each noise level, we record the percentage of data sets from which a stabilizing controller was found for $(A_{\text{true}}, B_{\text{true}})$ using the formulation (6.29). We display the results in the following table.

$\varepsilon = 0.5$	$\varepsilon = 1$	$\varepsilon = 1.5$	$\varepsilon = 2$	$\varepsilon = 2.2$	$\varepsilon = 2.4$
100%	96%	90%	82%	75%	73%

For $\varepsilon = 0.5$ we find a stabilizing controller in all 100 cases. When the noise level increases, the percentage of data sets for which the LMI (6.29) is feasible decreases. The interpretation is that by increasing the noise we enlarge the set of explaining systems $\Sigma_{\mathcal{D}}$. It thus becomes harder to simultaneously stabilize the systems in $\Sigma_{\mathcal{D}}$. Nonetheless, even for the larger noise level of $\varepsilon = 2.4$ we find a stabilizing controller in 73 out of the 100 data sets.

6.4 Noise-free versus noisy data

In the case of noise-free data, Theorem 6.5 gives a necessary and sufficient condition for informativity for stabilization. Moreover, in the case of noisy data, Theorem 6.25 provides a necessary and sufficient condition for informativity for quadratic stabilization. A natural question is now the following: what is the relation between these two theorems, and can the former be obtained as a special case from the latter? In this section, we will address these issues.

As in Section 6.1, consider the model class $\mathcal{M}$ of all noise free discrete-time linear input-state systems of the form

$$x(t+1) = Ax(t) + Bu(t)$$

where x is the n-dimensional state and u is the m-dimensional input. We assume that input-state data have been collected on the time interval $[0, T]$, leading to the data $\mathcal{D} := (U_-, X)$. We assume that these data are generated by the unknown system $(A_{\text{true}}, B_{\text{true}})$. The set of all systems in $\mathcal{M}$ that are consistent with the data is again denoted by $\Sigma_{\mathcal{D}}$. Since the unknown system is assumed to be consistent with the data, we have that $\Sigma_{\mathcal{D}}$ is nonempty.

As an alternative for Theorem 6.5 we will now prove that informativity for stabilization by state feedback in the noise free case can be characterized as follows.

Theorem 6.29. *The data (U_-, X) are informative for stabilization by state feedback if and only if there exist an $n \times n$ matrix $P > 0$, an $L \in \mathbb{R}^{m \times n}$, and a scalar $\beta > 0$ satisfying*

$$\begin{bmatrix} P - \beta I & 0 & 0 & 0 \\ 0 & -P & -L^\top & 0 \\ 0 & -L & 0 & L \\ 0 & 0 & L^\top & P \end{bmatrix} + \begin{bmatrix} X_+ \\ -X_- \\ -U_- \\ 0 \end{bmatrix} \begin{bmatrix} X_+ \\ -X_- \\ -U_- \\ 0 \end{bmatrix}^\top \geqslant 0. \tag{6.46}$$

Moreover, if P and L satisfy (6.46) then $K := LP^{-1}$ is a stabilizing feedback gain for all $(A, B) \in \Sigma_{\mathcal{D}}$.

The idea behind Theorem 6.29 is the following: in order to recover the noise-free case from Theorem (6.25) it is helpful to consider the noise model (3.18) as introduced in Section 3.4 with

$$\Phi = \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix}. \tag{6.47}$$

Indeed, this noise model implies that $W_- W_-^\top \leqslant 0$, i.e., $W_- = 0$ which corresponds exactly to the case in which the data are noise-free. We then see that

Theorem 6.29 bridges the exact and noisy case in the following sense. Note that the matrix on the right of (6.46) equals

$$-\begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \end{bmatrix}^\top,$$

which is nothing but a special case of the matrix on the right of (6.29) for the choices $\Phi_{11} = 0$, $\Phi_{12} = 0$ and $\Phi_{22} = -I$. Theorem 6.29 states that feasibility of the LMI (6.29) (with specific Φ) is necessary and sufficient for informativity for stabilization in the case of exact data. Moreover, in that case the assumption of a common Lyapunov function is not restrictive, i.e., informativity for stabilization is equivalent to informativity for quadratic stabilization in the case that w is absent.

Proof of Theorem 6.29. It follows from Theorem 6.4 that in the noise-free setting, if the data (U_-, X) are informative for stabilization and K is a suitable controller, then $A + BK = X_+X_-^\sharp$ for all $(A, B) \in \Sigma_\mathcal{D}$. In other words, the matrix $A + BK$ is the same for all systems (A, B) consistent with the data. This means that in the noise-free case, the data (U_-, X) are informative for stabilization *if and only if* they are informative for quadratic stabilization as in Definition 6.22 with the *specific* noise model (6.47). As such, the theorem follows readily from Theorem 6.25. □

Given the two equivalent conditions in Proposition 6.5 and Theorem 6.29 it is natural to question the relative merits of both approaches. First of all, we note that the LMI conditions in (6.5) and (6.46) are different in nature: the variable Θ in (6.5) has dimension $T \times n$ which depends on the time horizon of the experiment, while the dimensions of the variables P, L and β in (6.46) are independent of T. From a computational point of view, Theorem 6.29 may thus be preferred in cases where the time horizon T is large, for example if the inputs of the experiment are chosen to be persistently exciting, since this puts a lower bound $T \geqslant n + m + nm$ on the required number of samples (see (1.6) in Chapter 1).

On the other hand, in Chapter 12 it will be shown that for controllable pairs $(A_{\text{true}}, B_{\text{true}})$, the data (U_-, X) can be made informative for stabilization with at most $T = n + m$ samples, using an online input design method. In this case, the LMI (6.5) may be preferred since (6.5) has dimension $2n \times 2n$ which is smaller than the dimension $(3n + m) \times (3n + m)$ of (6.46).

6.5 Data-driven stabilization of Lur'e systems

In this section, we will consider the stabilization of a class of Lur'e systems, i.e., linear systems in feedback with a static nonlinearity. First, we will explain the classical problem of absolute stability analysis for such systems. Consider the Lur'e system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + E\varphi(y(t)) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{6.48}$$

where x is the n-dimensional state, u the m-dimensional input, y the p-dimensional output and $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the so-called *sector condition*

$$\varphi(y)(\varphi(y) - 2y) \leqslant 0 \quad \forall y \in \mathbb{R}. \tag{6.49}$$

We note that more general sector conditions can be transformed to (6.49) by means of loop transformations. The real matrices A, B, C, D and E are of appropriate dimensions. Suppose that we apply a state feedback controller $u = Kx$ resulting in

$$\begin{aligned} x(t+1) &= (A + BK)x(t) + E\varphi(y(t)) \\ y(t) &= (C + DK)x(t). \end{aligned} \tag{6.50}$$

For systems of the form (6.50), a problem with a rich history is that of *absolute stability*, i.e. global asymptotic stability of the equilibrium point 0 of (6.50) *for all* continuous sector-bounded nonlinearities satisfying (6.49). We focus on showing absolute stability of (6.50) by means of a quadratic Lyapunov function $V(x) := x^\top Px$ where $P > 0$. To this end, it is sufficient to show that the Lyapunov inequality $V(x(t+1)) < V(x(t))$ holds for all continuous φ satisfying (6.49) and all nonzero $x(t)$ and resulting $x(t+1)$ satisfying (6.50). Let $A_K := A + BK$ and $C_K := C + DK$.

The Lyapunov inequality holds if

$$(A_K x + Ew)^\top P(A_K x + Ew) - x^\top Px < 0$$

for all $w \in \mathbb{R}$ and nonzero $x \in \mathbb{R}^n$ satisfying $w(w - 2C_K x) \leqslant 0$. Equivalently,

$$\begin{bmatrix} x \\ w \end{bmatrix}^\top \begin{bmatrix} P - A_K^\top PA_K & -A_K^\top PE \\ -E^\top PA_K & -E^\top PE \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} > 0 \tag{6.51}$$

for all $w \in \mathbb{R}$ and nonzero $x \in \mathbb{R}^n$ satisfying

$$\begin{bmatrix} x \\ w \end{bmatrix}^\top \begin{bmatrix} 0 & C_K^\top \\ C_K & -1 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \geqslant 0. \tag{6.52}$$

Since (6.52) is not satisfied when $x = 0$ and $w \neq 0$, the latter statement is equivalent to (6.51) being satisfied for all nonzero (x, w) satisfying (6.52). If

$C_K \neq 0$ then the inequality (6.52) is strictly feasible. Moreover, if $C_K = 0$ then the inequality (6.52) reduces to the equality

$$\begin{bmatrix} x \\ w \end{bmatrix}^\top \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0.$$

Thus, by applying the S-lemma (Proposition A.12) in the former case and Finsler's lemma (Proposition A.13) in the latter, we conclude that (6.51) is satisfied for all nonzero (x, w) satisfying (6.52) if and only if

$$\begin{bmatrix} P - A_K^\top P A_K & -A_K^\top P E \\ -E^\top P A_K & -E^\top P E \end{bmatrix} - \alpha \begin{bmatrix} 0 & C_K^\top \\ C_K & -1 \end{bmatrix} > 0 \tag{6.53}$$

for some scalar $\alpha \geqslant 0$. Proving absolute stability of (6.50) by a quadratic Lyapunov function thus boils down to finding $P > 0$ and $\alpha \geqslant 0$ such that (6.53) holds. Since $-E^\top P E \leqslant 0$, we must have $\alpha > 0$ and therefore, by homogeneity, we can even get rid of α and look for $P > 0$ satisfying

$$\begin{bmatrix} P - A_K^\top P A_K & -A_K^\top P E - C_K^\top \\ -E^\top P A_K - C_K & 1 - E^\top P E \end{bmatrix} > 0. \tag{6.54}$$

Next, we focus on data-based stabilization of Lur'e systems. Consider the system

$$\begin{aligned} x(t+1) &= A_{\text{true}} x(t) + B_{\text{true}} u(t) + E\varphi(y(t)) + w(t) \\ y(t) &= C_{\text{true}} x(t) + D_{\text{true}} u(t) + v(t) \end{aligned} \tag{6.55}$$

where $A_{\text{true}}, B_{\text{true}}, C_{\text{true}}$ and D_{true} are unknown real matrices and the matrix E is known. The signals w and v are process and measurement noise terms that are unknown. We obtain state and input measurements from (6.55), collected in the matrices X and U_- as before, in addition to corresponding measurements of the form

$$\begin{aligned} Y_- &:= Y_{[0,T-1]} \\ F_- &:= \begin{bmatrix} \varphi(y(0)) & \varphi(y(1)) & \cdots & \varphi(y(T-1)) \end{bmatrix}. \end{aligned} \tag{6.56}$$

This means that the data are given by $\mathcal{D} = (U_-, F_-, X, Y_-)$. During the experiment, the noise samples

$$W_- := \begin{bmatrix} W_{[0,T-1]} \\ V_{[0,T-1]} \end{bmatrix}$$

are assumed to satisfy $W_-^\top \in \mathcal{Z}_T(\Phi)$ for some known matrix $\Phi \in \mathbf{\Pi}_{n+1,T}$. If we define X_+ and X_- as before then all systems (A, B, C, D) consistent with the data are given by the set $\Sigma_{\mathcal{D}}$ defined by

$$\Sigma_{\mathcal{D}} := \left\{ (A, B, C, D) \mid \begin{bmatrix} X_+ - EF_- \\ Y_- \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = W_- \text{ for some } W_-^\top \in \mathcal{Z}_T(\Phi) \right\}.$$

This leads to the following definition of informative data.

Definition 6.30. Let $\Phi \in \mathbf{\Pi}_{n+1,T}$. Suppose that the data (U_-, F_-, X, Y_-) have been generated by (6.55) for some noise matrix $W_-^\top \in \mathcal{Z}_T(\Phi)$. Then (U_-, F_-, X, Y_-) are called *informative for absolute quadratic stabilization* if there exist an $n \times n$ matrix $P > 0$ and a $K \in \mathbb{R}^{m \times n}$ such that (6.54) holds for all $(A, B, C, D) \in \Sigma_\mathcal{D}$.

Next, we state the following theorem that gives a necessary and sufficient condition for informativity for absolute quadratic stabilization.

Theorem 6.31. *Let $\Phi \in \mathbf{\Pi}_{n+1,T}$ and consider the data (U_-, F_-, X, Y_-), generated by (6.55) for some $W_-^\top \in \mathcal{Z}_T(\Phi)$. Then (U_-, F_-, X, Y_-) are informative for absolute quadratic stabilization if and only if there exist an $n \times n$ matrix $Q > 0$, an $L \in \mathbb{R}^{m \times n}$ and scalars $\alpha \geqslant 0$ and $\beta > 0$ such that*

$$\begin{bmatrix} Q - \beta I & -E & 0 & 0 & 0 \\ -E^\top & 1 - \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Q \\ 0 & 0 & 0 & 0 & L \\ 0 & 0 & Q & L^\top & Q \end{bmatrix} - \alpha \begin{bmatrix} I & 0 & X_+ - EF_- \\ 0 & 1 & Y_- \\ 0 & 0 & -X_- \\ 0 & 0 & -U_- \\ 0 & 0 & 0 \end{bmatrix} \Phi \begin{bmatrix} I & 0 & X_+ - EF_- \\ 0 & 1 & Y_- \\ 0 & 0 & -X_- \\ 0 & 0 & -U_- \\ 0 & 0 & 0 \end{bmatrix}^\top \geqslant 0.$$

In this case, the feedback gain $K := LQ^{-1}$ is such that (6.50) is absolutely stable for all $(A, B, C, D) \in \Sigma_\mathcal{D}$.

The proof follows similar lines as that of Theorem 6.25. It uses a dualization step (see Lemma A.3) on the inequality (6.54) and relies on Corollary A.23 using the relation $Q = P^{-1}$ between P and Q.

6.6 Notes and references

For exact input-state data, informativity for stabilization was introduced in [175]. This paper also provided the characterizations in Theorems 6.4 and 6.5. It is noteworthy that the linear matrix inequality condition (6.5) was considered before in [44] under the additional assumptions that the input sequence is persistently exciting of order $n + 1$ and the true system is controllable. Under these assumptions, the true system is the only system consistent with the data. However, as we have demonstrated in Example 6.6, there are situations in which the set of consistent systems is not a singleton, but there does exist a stabilizing controller for all systems consistent with the data. In this sense, the notion of informativity for stabilization is weaker than the notion of informativity for identification as studied in Section 3.1. The concept of informativity for deadbeat control and the corresponding characterization in Theorem 6.8 were introduced in [175].

Subsection 6.2.1 on stabilization by dynamic output feedback using exact input-(state)-output data is based on [175]. In order to obtain a sufficient condition for the informativity of input-output data in Corollary 6.21, we have relied on an idea from subspace identification. In particular, Theorem 6.20 shows how to obtain a state sequence from input-output data, assuming that a certain rank condition on data Hankel matrices is satisfied. This result is a reformulation of [115, Thm. 3]. For further details on subspace identification see [165, 178].

In the context of noisy input-state data, several sufficient conditions for informativity for quadratic stabilization can be found in the literature, see [44, Thm. 2] and [18, Thm. 4]. The paper [169] provided necessary and sufficient conditions by making use of a matrix version of the S-lemma (see Section A.3). These conditions were formulated under the assumption of a so-called generalized Slater condition on a matrix constructed from the noise model and the data. This assumption was removed in [168]. Theorem 6.25 of this book is based on the latter paper. For bounded sets of consistent systems, data-driven stabilization was also approached from the perspective of Petersen's lemma in [23]. The relation between Petersen's lemma and the matrix S-lemma was further clarified in [168], where it was shown that the former can be obtained as a special case of the latter.

In Section 6.4 we have clarified the relation between data-driven stabilization using noise-free data and quadratic stabilization using noisy data. This section is based on the paper [167].

The problem of absolute stability of Lur'e systems dates back to work by Lur'e and Postnikov [100] and Popov [131]. The approach to stabilize Lur'e systems in Section 6.5 mimics the continuous-time setting of [26, Ch. 5]. The data-driven stabilization of Lur'e systems in this section is based on [168]. We also refer to [26] for a discussion on the topic of loop transformations.

7

LQR control design from data

In this chapter we study the data-driven linear quadratic regulator (LQR) problem. We will consider both the optimal as well as the suboptimal version of the the problem, in Sections 7.1 and 7.2, respectively. For these problems, we will present conditions under which a given data set is informative for control design. We will also establish design methods for obtaining suitable controllers from the data, in terms of data based linear matrix inequalities.

7.1 The data-driven optimal LQR problem

Consider the linear system

$$x(t+1) = A_{\text{true}}x(t) + B_{\text{true}}u(t) \tag{7.1}$$

where A_{true} and B_{true} are unknown matrices of given (known) dimensions $n \times n$ and $n \times m$, and where x is the n-dimensional state and u the m-dimensional input.

Suppose that we want to quantify the performance of the system using a given cost functional $J(x_0, u)$. In the context of linear quadratic optimal control, this cost functional is a quadratic functional of the state trajectory x and the input u. The optimal linear quadratic regulator problem is then to find, for each initial state x_0 of the system, an optimal input, i.e. an input that minimizes the cost functional. In the situation that the system matrices A_{true} and B_{true} are known, minimization of this cost functional can be performed using existing methods. In a data-driven context however, these system matrices are not known, so the existing methods are not applicable. Instead, data on the system should be used to find optimal inputs.

Before embarking on this data-driven version of the optimal linear quadratic regulator problem, we will first briefly review some of the basics of discrete-time linear quadratic optimal control. In the sequel, we will use the abbreviation 'LQR' for 'linear quadratic regulator'.

Consider, in general, the discrete-time linear system

$$x(t+1) = Ax(t) + Bu(t) \tag{7.2}$$

with, as before, x the n-dimensional state and u is the m-dimensional input. For any initial state x_0, let $x_{x_0,u}$ be the state sequence of (7.2) resulting from the input u and initial condition $x(0) = x_0$. We omit the subscript and simply write x whenever the dependence on x_0 and u is clear from the context.

Associated to system (7.2), we define the quadratic cost functional

$$J(x_0, u) = \sum_{t=0}^{\infty} x^\top(t)Qx(t) + u^\top(t)Ru(t) \tag{7.3}$$

where $Q \in \mathbb{S}^n$ is positive semidefinite and $R \in \mathbb{S}^m$ is positive definite. Then, the optimal LQR problem is the following:

Problem 7.1. Determine for every initial state x_0 an input u^*, such that $\lim_{t\to\infty} x_{x_0,u^*}(t) = 0$, and the cost functional $J(x_0, u)$ is minimized under this constraint.

Such an input u^* is called optimal for the given x_0. Of course, an optimal input does not necessarily exist for all x_0. We say that the optimal LQR problem is *solvable* for (A, B, Q, R) if for every x_0 there exists an input u^* such that

(a) The cost $J(x_0, u^*)$ is finite.

(b) The limit $\lim_{t\to\infty} x_{x_0,u^*}(t) = 0$.

(c) The input u^* minimizes the cost functional, i.e.,

$$J(x_0, u^*) \leqslant J(x_0, \bar{u})$$

for all $\bar{u}$ such that $\lim_{t\to\infty} x_{x_0,\bar{u}}(t) = 0$.

In the sequel, we will require the notion of observable eigenvalue. An eigenvalue λ of A is called (Q, A)-observable if

$$\operatorname{rank} \begin{pmatrix} A - \lambda I \\ Q \end{pmatrix} = n.$$

The following theorem provides necessary and sufficient conditions for the solvability of the optimal LQR problem for (A, B, Q, R). This theorem is the discrete-time analogue of the continuous-time case stated in [160, Thm. 10.18].

Theorem 7.2. *Let $Q \geqslant 0$ and $R > 0$. Then the following statements hold:*

(a) *If (A, B) is stabilizable, there exists a unique largest real symmetric solution P^+ to the discrete-time algebraic Riccati equation (DARE)*

$$P = A^\top PA - A^\top PB(R + B^\top PB)^{-1}B^\top PA + Q, \tag{7.4}$$

in the sense that $P^+ \geqslant P$ for every real symmetric P satisfying (7.4). The matrix P^+ is positive semidefinite.

(b) *If, in addition to stabilizability of (A, B), every eigenvalue of A on the unit circle is (Q, A)-observable then for every x_0 a unique optimal input u^* exists. Furthermore, this input sequence is generated by the feedback law $u = Kx$, where*

$$K := -(R + B^\top P^+ B)^{-1} B^\top P^+ A. \tag{7.5}$$

Moreover, the matrix $A + BK$ is stable.

(c) *In fact, the optimal LQR problem is solvable for (A, B, Q, R) if and only if (A, B) is stabilizable and every eigenvalue of A on the unit circle is (Q, A)-observable.*

If the optimal LQR problem is solvable for (A, B, Q, R), we say that the matrix K given by (7.5) is the optimal feedback gain for (A, B, Q, R).

We will now turn to the data-driven version of the optimal LQR problem. For given state and input dimensions n and m, we consider the model class $\mathcal{M}$ of all discrete-time linear input-state systems of the form (7.2). Assume we have input-state samples on the time interval $[0, T]$, generated by the true system (7.1), leading to data $\mathcal{D} := (U_-, X)$ as given by (2.1). As before, the set $\Sigma_\mathcal{D}$ of all systems in $\mathcal{M}$ that are consistent with the data is then equal to $\Sigma_{(U_-,X)}$ defined by

$$\Sigma_{(U_-,X)} := \left\{ (A, B) \in \mathcal{M} \mid X_+ = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right\}. \tag{7.6}$$

Note that the true system $(A_{\text{true}}, B_{\text{true}})$ is a member of $\Sigma_{(U_-,X)}$. In the context of the optimal LQR problem the control objective $\mathcal{O}$ is: 'the system must be controlled using the optimal feedback gain'. In order to formalize this, we introduce the following notation. For any given K, let $\Sigma_K^{Q,R}$ denote the set of all systems of the form (7.2) for which K is the optimal feedback gain corresponding to Q and R, that is,

$$\Sigma_K^{Q,R} := \{ (A, B) \in \mathcal{M} \mid K \text{ is the optimal gain for } (A, B, Q, R) \}.$$

This gives rise to yet another notion of informativity in line with Definition 2.4. Indeed, informativity requires the existence of a feedback gain that is optimal for all systems consistent with the data.

Definition 7.3. Given matrices Q and R, we say that the data (U_-, X) are *informative for optimal linear quadratic regulation* if the optimal LQR problem is solvable for all $(A, B) \in \Sigma_{(U_-,X)}$ and there exists K such that $\Sigma_{(U_-,X)} \subseteq \Sigma_K^{Q,R}$.

In order to provide necessary and sufficient conditions for the corresponding informativity problem, we need the following auxiliary lemma.

Lemma 7.4. *Let $Q \geqslant 0$ and $R > 0$. Suppose the data (U_-, X) are informative for optimal linear quadratic regulation. Let K be such that $\Sigma_{(U_-,X)} \subseteq \Sigma_K^{Q,R}$. Then, there exist a square matrix M and a matrix $P^+ \geqslant 0$ such that for all $(A, B) \in \Sigma_{(U_-,X)}$*

$$M = A + BK \tag{7.7}$$

$$P^+ = A^\top P^+ A - A^\top P^+ B(R + B^\top P^+ B)^{-1} B^\top P^+ A + Q \tag{7.8}$$

$$P^+ - M^\top P^+ M = K^\top R K + Q \tag{7.9}$$

$$K = -(R + B^\top P^+ B)^{-1} B^\top P^+ A. \tag{7.10}$$

Proof. Since the data (U_-, X) are informative for optimal linear quadratic regulation, $A + BK$ is stable for every $(A, B) \in \Sigma_{(U_-,X)}$. By Lemma 6.3, this implies that $A_0 + B_0 K = 0$ for all $(A_0, B_0) \in \Sigma^0_{(U_-,X)}$. Thus, there exists M such that $M = A + BK$ for all $(A, B) \in \Sigma_{(U_-,X)}$. For the rest, note that Theorem 7.2 implies that for every $(A, B) \in \Sigma_{(U_-,X)}$ there exists $P^+_{(A,B)}$ satisfying the DARE

$$P^+_{(A,B)} = A^\top P^+_{(A,B)} A - A^\top P^+_{(A,B)} B(R + B^\top P^+_{(A,B)} B)^{-1} B^\top P^+_{(A,B)} A + Q \tag{7.11}$$

such that

$$K = -(R + B^\top P^+_{(A,B)} B)^{-1} B^\top P^+_{(A,B)} A. \tag{7.12}$$

It is important to note that, although K is independent of the choice of (A, B), the matrix $P^+_{(A,B)}$ might depend on (A, B). We will, however, show that also $P^+_{(A,B)}$ is independent of the choice of (A, B).

By rewriting (7.11), we see that

$$P^+_{(A,B)} - M^\top P^+_{(A,B)} M = K^\top R K + Q. \tag{7.13}$$

Since M is stable, $P^+_{(A,B)}$ is the unique solution to the discrete-time Lyapunov equation (7.13). Moreover, since M and K do not depend on the choice of $(A, B) \in \Sigma_{(U_-,X)}$, it indeed follows that $P^+_{(A,B)}$ does not depend on (A, B). It follows from (7.11)–(7.13) that $P^+ := P^+_{(A,B)}$ satisfies (7.8)–(7.10). ☐

Recall from Definition 2.12 that the data (U_-, X) are called informative for identification if $\Sigma_{(U_-,X)}$ contains exactly one element, equivalently $\Sigma_{(U_-,X)} = \{(A_{\text{true}}, B_{\text{true}})\}$ (see also Theorem 3.1). Then the following theorem solves the informativity problem for optimal linear quadratic regulation.

Theorem 7.5. *Let $Q \geqslant 0$ and $R > 0$. Then the data (U_-, X) are informative for optimal linear quadratic regulation if and only if at least one of the following two conditions hold:*

(a) *The data* (U_-, X) *are informative for identification and the optimal LQR problem is solvable for* $(A_{\text{true}}, B_{\text{true}}, Q, R)$*. In this case, the optimal feedback gain* K *is given by* (7.5) *where* P^+ *is the largest real symmetric solution to* (7.4) *with* $A = A_{\text{true}}$ *and* $B = B_{\text{true}}$*.*

(b) *For all* $(A, B) \in \Sigma_{(U_-,X)}$ *we have* $A = A_{\text{true}}$*. Moreover,* A_{true} *is stable and* $QA_{\text{true}} = 0$*. In this case the optimal feedback gain is given by* $K = 0$*.*

Remark 7.6. Condition (b) of Theorem 7.5 is a pathological case in which A is stable and $QA = 0$ for all matrices A that are consistent with the data. In this case, if the input function is chosen as $u = 0$ then $x(t) \in \operatorname{im} A$ for all $t > 0$, so $Qx(t) = 0$ for all $t > 0$. Additionally, since A is stable, this shows that the optimal input is equal to $u^* = 0$. If we set aside the pathological case (b), the main message of Theorem 7.5 is the following: if the data are informative for optimal linear quadratic regulation they are also informative for system identification.

Proof of Theorem 7.5. We first prove the 'if' part. Sufficiency of the condition (a) readily follows from Theorem 7.2. To prove the sufficiency of the condition (b), assume that the matrix A is stable and $QA = 0$ for all $(A, B) \in \Sigma_{(U_-,X)}$. By the discussion in Remark 7.6 implies that $u^* = 0$ for all $(A, B) \in \Sigma_{(U_-,X)}$. Hence, for $K = 0$ we have $\Sigma_{(U_-,X)} \subseteq \Sigma_K^{Q,R}$, i.e., the data are informative for linear quadratic regulation.

To prove the 'only if' part, suppose that the data (U_-, X) are informative for optimal linear quadratic regulation. From Lemma 7.4, we know that there exist M and P^+ satisfying (7.7)–(7.10) for all $(A, B) \in \Sigma_{(U_-,X)}$. By substituting (7.10) into (7.8) and using (7.7), we obtain

$$A^\top P^+ M = P^+ - Q. \tag{7.14}$$

In addition, it follows from (7.10) that $-(R + B^\top P^+ B)K = B^\top P^+ A$. By using (7.7), we have

$$B^\top P^+ M = -RK. \tag{7.15}$$

Since (7.14) and (7.15) hold for all $(A, B) \in \Sigma_{(U_-,X)}$, we have that

$$\begin{bmatrix} A_0^\top \\ B_0^\top \end{bmatrix} P^+ M = 0$$

for all $(A_0, B_0) \in \Sigma^0_{(U_-,X)}$. Note that $(FA_0, FB_0) \in \Sigma^0_{(U_-,X)}$ for all $F \in \mathbb{R}^{n\times n}$ whenever $(A_0, B_0) \in \Sigma^0_{(U_-,X)}$. This means that

$$\begin{bmatrix} A_0^\top \\ B_0^\top \end{bmatrix} F^\top P^+ M = 0$$

for all $F \in \mathbb{R}^{n\times n}$. Therefore, either $\begin{bmatrix} A_0 & B_0 \end{bmatrix} = 0$ for all $(A_0, B_0) \in \Sigma^0_{(U_-,X)}$ or $P^+M = 0$. The former is equivalent to $\Sigma^0_{(U_-,X)} = \{0\}$. In this case, we see that the data (U_-, X) are informative for identification, equivalently $\Sigma_{(U_-,X)} = \{(A_{\text{true}}, B_{\text{true}})\}$, and the optimal LQR problem is solvable for $(A_{\text{true}}, B_{\text{true}}, Q, R)$. Therefore, condition (a) holds. On the other hand, if $P^+M = 0$ then we have

$$\begin{aligned} 0 = P^+M = P^+(A + BK) & \\ &= P^+\big(A - B(R + B^\top P^+B)^{-1}B^\top P^+A\big) \\ &= \big(I - P^+B(R + B^\top P^+B)^{-1}B^\top\big)P^+A. \end{aligned}$$

for all $(A, B) \in \Sigma_{(U_-,X)}$. From the identity

$$(I + P^+BR^{-1}B^\top)^{-1} = I - P^+B(R + B^\top P^+B)^{-1}B^\top$$

we see that $P^+A = 0$ for all $(A, B) \in \Sigma_{(U_-,X)}$. Then, it follows from (7.10) that $K = 0$. Since $A_0 + B_0K = 0$ for all $(A_0, B_0) \in \Sigma^0_{(U_-,X)}$ due to Lemma 6.3, we see that A_0 must be zero. Hence, we have $A = A_{\text{true}}$ for all $(A, B) \in \Sigma_{(U_-,X)}$ and A_{true} is stable. Moreover, it follows from (7.14) that $P^+ = Q$. Therefore, $QA_{\text{true}} = 0$. In other words, condition (b) is satisfied, which proves the theorem. □

Theorem 7.5 gives necessary and sufficient conditions under which the data are informative for optimal linear quadratic regulation. However, it might not be directly clear how these conditions can be verified given input-state data. Therefore, in what follows we rephrase the conditions of Theorem 7.5 in terms of the data matrices X and U_-.

Theorem 7.7. *Let $Q \geqslant 0$ and $R > 0$. Then the data (U_-, X) are informative for optimal linear quadratic regulation if and only if at least one of the following two conditions hold:*

(a) *The data (U_-, X) are informative for identification. Equivalently, there exists $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ such that (6.14) holds. Moreover, the optimal LQR problem is solvable for $(A_{\text{true}}, B_{\text{true}}, Q, R)$, where $A_{\text{true}} = X_+V_1$ and $B_{\text{true}} = X_+V_2$.*

(b) *There exists $\Theta \in \mathbb{R}^{T\times n}$ such that $X_-\Theta = (X_-\Theta)^\top$, $U_-\Theta = 0$,*

$$\begin{bmatrix} X_-\Theta & X_+\Theta \\ \Theta^\top X_+^\top & X_-\Theta \end{bmatrix} > 0 \tag{7.16}$$

and $QX_+\Theta = 0$.

Proof. The equivalence of condition (a) of Theorem 7.5 and condition (a) of Theorem 7.7 is obvious. It remains to be shown that condition (b) of Theorem 7.5 and condition (b) of Theorem 7.7 are equivalent as well. To this end, suppose that there exists a matrix $\Theta \in \mathbb{R}^{T\times n}$ such that the conditions of (b) holds. By Theorem 6.5, we have that $A+BK$ is stable for $K=0$ for all $(A,B) \in \Sigma_{(U_-,X)}$, that is, A is stable for all $(A,B) \in \Sigma_{(U_-,X)}$. In addition, note that

$$QX_+\Theta(X_-\Theta)^{-1} = Q\begin{bmatrix} A & B \end{bmatrix}\begin{bmatrix} X_- \\ U_- \end{bmatrix}\Theta(X_-\Theta)^{-1} = QA \tag{7.17}$$

for all $(A,B) \in \Sigma_{(U_-,X)}$. This shows that $QA=0$ and therefore that condition (b) of Theorem 7.5 holds. Conversely, suppose that A is stable and $QA=0$ for all $(A,B) \in \Sigma_{(U_-,X)}$. This implies that $K=0$ is a stabilizing controller for all $(A,B) \in \Sigma_{(U_-,X)}$. By Theorem 6.5, there exists a matrix $\Theta \in \mathbb{R}^{T\times n}$ satisfying the first three conditions of (b). Finally, it follows from $QA=0$ and (7.17) that Θ also satisfies the fourth equation of (b). This proves the theorem. ☐

7.1.1 From data to optimal LQR gain

In this subsection we will devise a method to compute the optimal LQR feedback gain K directly from the data. For this, we will employ ideas from the study of Riccati inequalities.

The following theorem asserts that P^+ as in Lemma 7.4 can be found as the unique solution to an optimization problem involving only the data. Furthermore, the optimal feedback gain K can subsequently be found by solving a set of linear equations. Recall that, for a given square matrix M, $\operatorname{tr}(M)$ denotes the trace of M.

Theorem 7.8. *Let $Q \geqslant 0$ and $R > 0$. Suppose that the data (U_-,X) are informative for optimal linear quadratic regulation. Consider the linear transformation $P \mapsto \mathcal{L}(P)$ defined by*

$$\mathcal{L}(P) := X_-^\top P X_- - X_+^\top P X_+ - X_-^\top Q X_- - U_-^\top R U_-.$$

Let P^+ be as in Lemma 7.4. The following statements hold:

(a) *The matrix P^+ is equal to the unique solution to the optimization problem*

$$\begin{aligned} &\textit{maximize} \quad \operatorname{tr}(P) \\ &\textit{subject to} \quad P \geqslant 0 \quad \textit{and} \quad \mathcal{L}(P) \leqslant 0. \end{aligned}$$

(b) *There exists a right inverse $X_-^\sharp$ of X_- such that*

$$\mathcal{L}(P^+)X_-^\sharp = 0. \tag{7.18}$$

Moreover, if $X_-^\sharp$ satisfies (7.18), *then the optimal feedback gain is given by* $K = U_- X_-^\sharp$.

Remark 7.9. From a design viewpoint, the optimal feedback gain K can be found in the following way. First solve the semidefinite program in Theorem 7.8((a)). Subsequently, compute a solution $X_-^\sharp$ to the linear equations $X_- X_-^\sharp = I$ and (7.18). Then, the optimal feedback gain is given by $K = U_- X_-^\sharp$.

Proof of Theorem 7.8. We begin with proving statement (i). We claim that the following implication holds:

$$P \geqslant 0 \;\text{ and }\; \mathcal{L}(P) \leqslant 0 \implies P^+ \geqslant P. \tag{7.19}$$

To prove this claim, let P be such that $P \geqslant 0$ and $\mathcal{L}(P) \leqslant 0$. Since the data are informative for optimal linear quadratic regulation, they are also informative for stabilization by state feedback. Therefore, the optimal feedback gain K satisfies

$$\operatorname{im} \begin{bmatrix} I \\ K \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} X_- \\ U_- \end{bmatrix}$$

due to Lemma 6.3. In addition, note that

$$\mathcal{L}(P) = \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top \begin{bmatrix} P - A^\top P A - Q & -A^\top P B \\ -B^\top P A & -(R + B^\top P B) \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}$$

for all $(A, B) \in \Sigma_{(U_-,X)}$. Therefore

$$\begin{bmatrix} I \\ K \end{bmatrix}^\top \begin{bmatrix} P - A^\top P A - Q & -A^\top P B \\ -B^\top P A & -(R + B^\top P B) \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} \leqslant 0$$

for all $(A, B) \in \Sigma_{(U_-,X)}$. This yields

$$P - M^\top P M \leqslant K^\top R K + Q$$

where M is as in Lemma 7.4. By subtracting this from (7.9), we obtain

$$(P^+ - P) - M^\top (P^+ - P) M \geqslant 0.$$

Since M is stable, this discrete-time Lyapunov inequality implies that $P^+ - P \geqslant 0$ and hence $P^+ \geqslant P$. This proves the claim (7.19).

Note that $R + B^\top P^+ B$ is positive definite. Then, it follows from (7.8) that for all $(A, B) \in \Sigma_{(U_-,X)}$ we have

$$\begin{bmatrix} P^+ - A^\top P^+ A - Q & -A^\top P^+ B \\ -B^\top P^+ A & -(R + B^\top P^+ B) \end{bmatrix} \leqslant 0$$

via a Schur complement argument. Hence, $\mathcal{L}(P^+) \leqslant 0$. Since $P^+ \geqslant P$, we have $\operatorname{tr}(P^+) \geqslant \operatorname{tr}(P)$. Together with (7.19), this shows that P^+ is a solution to the optimization problem stated in the theorem.

Next, we prove uniqueness. Let $\bar{P}$ be any solution of the optimization problem. Then, we have that $\bar{P} \geqslant 0$, $\mathcal{L}(\bar{P}) \leqslant 0$, and $\operatorname{tr}(\bar{P}) = \operatorname{tr}(P^+)$. From (7.19), we see that $P^+ \geqslant \bar{P}$. In particular, this implies that $(P^+)_{ii} \geqslant \bar{P}_{ii}$ for all i. Together with $\operatorname{tr}(\bar{P}) = \operatorname{tr}(P^+)$, this implies that $(P^+)_{ii} = \bar{P}_{ii}$ for all i. Now, for any i and j, we have

$$(e_i - e_j)^\top P^+ (e_i - e_j) \geqslant (e_i - e_j)^\top \bar{P} (e_i - e_j) \text{ and}$$
$$(e_i + e_j)^\top P^+ (e_i + e_j) \geqslant (e_i + e_j)^\top \bar{P} (e_i + e_j)$$

where e_i denotes the i-th standard basis vector. This leads to $(P^+)_{ij} \leqslant \bar{P}_{ij}$ and $(P^+)_{ij} \geqslant \bar{P}_{ij}$, respectively. We conclude that $(P^+)_{ij} = \bar{P}_{ij}$ for all i,j. This proves uniqueness.

Finally, we prove the statement (ii). It follows from (7.8) and (7.10) that

$$\mathcal{L}(P^+) = -\left(U_- - KX_-\right)^\top \left(R + B^\top P^+ B\right)\left(U_- - KX_-\right). \tag{7.20}$$

The optimal feedback K is stabilizing, therefore it follows from Theorem 6.4 that K can be written as $K = U_-\Gamma$, where Γ is some right inverse of X_-. Note that this implies the existence of a right inverse $X_-^\sharp$ of X_- satisfying (7.18). Indeed, $X_-^\sharp := \Gamma$ is such a matrix by (7.20). Moreover, if $X_-^\sharp$ is a right inverse of X_- satisfying (7.18) then $(U_- - KX_-)X_-^\sharp = 0$ by (7.20) and positive definiteness of R. We conclude that the optimal feedback gain is equal to $K = U_- X_-^\sharp$, which proves the second statement. □

7.2 The data-driven suboptimal LQR problem

Again assume that our true (but unknown) system is given by (7.1) and that we want to quantify the performance of the system using a given quadratic cost functional (7.3). Whereas the optimal linear quadratic regulator problem is to find, for each initial state x_0 of the system, an optimal input sequence, i.e. an input sequence that minimizes the cost functional, the *suboptimal* linear quadratic regulator problem aims at finding an input such that its cost does not exceed a given tolerance. In the situation that the system matrices A_{true} and B_{true} are known, methods for computing such suboptimal inputs exist. In the data-driven version of the problem, only data on the true system are available to compute suboptimal inputs.

Before considering this data driven version of the suboptimal linear quadratic regulator problem, we will first briefly review some basics on the discrete-time suboptimal linear quadratic regulator problem. Again, in the sequel we will use the abbreviation 'LQR' for 'linear quadratic regulator'.

Consider the linear system (7.2). Let x_0 be a given initial state. Let (7.3) be a given quadratic cost functional, where $Q \geqslant 0$ and $R > 0$ are real matrices. Whenever the input function u results from a state feedback law $u = Kx$, we will write $J(x_0, K)$ instead of $J(x_0, u)$.

The suboptimal LQR problem is now formulated as follows. Given $x_0 \in \mathbb{R}^n$ and a tolerance $\gamma > 0$, find (if it exists) a feedback law $u = Kx$ such that $A + BK$ is stable, and the cost satisfies $J(x_0, K) < \gamma$. Such a K is called a *suboptimal feedback gain* for the system (7.2) with cost functional (7.3). The following proposition gives necessary and sufficient conditions under which a given matrix K is a suboptimal feedback gain.

Proposition 7.10. *Let $x_0 \in \mathbb{R}^n$ and $\gamma > 0$. The matrix K is a suboptimal feedback gain if and only if there exists a matrix $P > 0$ such that*

$$(A + BK)^\top P(A + BK) - P + Q + K^\top RK < 0 \tag{7.21}$$

$$x_0^\top P x_0 < \gamma. \tag{7.22}$$

To prove this proposition, we make use of the following lemma on the solution of Lyapunov equations. A proof of this lemma can be found in [12, Thm. 4.50].

Lemma 7.11. *Let $A \in \mathbb{R}^{n\times n}$ be stable. For any matrix $Q \in \mathbb{S}^n$, there exists a unique solution $P \in \mathbb{S}^n$ to the Lyapunov equation*

$$P - A^\top PA = Q. \tag{7.23}$$

Moreover, this solution P is given by $P = \sum_{t=0}^{\infty} (A^\top)^t Q A^t$.

Proof of Proposition 7.10. Assume that there exists a matrix $P > 0$ such that (7.21) and (7.22) hold. Since $Q \geqslant 0$ and $R > 0$, the inequality (7.21) implies that

$$(A + BK)^\top P(A + BK) - P < 0.$$

We conclude that $A+BK$ is stable. Thus, by Lemma 7.11 there exists a unique solution $\hat{P} \in \mathbb{S}^n$ to the Lyapunov equation

$$\hat{P} - (A + BK)^\top \hat{P}(A + BK) = Q + K^\top RK. \tag{7.24}$$

Moreover, we have that

$$\hat{P} = \sum_{t=0}^{\infty} ((A + BK)^\top)^t (Q + K^\top RK)(A + BK)^t. \tag{7.25}$$

Therefore, we can express the cost $J(x_0, K)$ in (7.3) in terms of $\hat{P}$ as

$$J(x_0, K) = x_0^\top \hat{P} x_0.$$

By adding (7.21) to (7.24) we obtain

$$\hat{P} - P - (A + BK)^\top (\hat{P} - P)(A + BK) = N$$

for some matrix $N < 0$. Invoking Lemma 7.11 once again, we conclude that

$$\hat{P} - P = \sum_{t=0}^{\infty} ((A + BK)^\top)^t N (A + BK)^t \leqslant 0.$$

Therefore, $J(x_0, K) = x_0^\top \hat{P} x_0 \leqslant x_0^\top P x_0 < \gamma$. We conclude that K is a suboptimal feedback gain.

Conversely, suppose that K is a suboptimal feedback gain. This means that $A + BK$ is stable and $J(x_0, K) = x_0^\top \hat{P} x_0 < \gamma$, where $\hat{P} \in \mathbb{S}^n$ is the unique solution to (7.24). Since $Q \geqslant 0$ and $R > 0$, we have that $\hat{P} \geqslant 0$. By Lemma 7.11 there exists a unique solution $P \in \mathbb{S}^n$ to the Lyapunov equation

$$P - (A + BK)^\top P (A + BK) = I. \tag{7.26}$$

Moreover, since $P = \sum_{t=0}^{\infty} ((A + BK)^\top)^t (A + BK)^t$, we have that $P > 0$. It follows from (7.26) that εP satisfies

$$\varepsilon P - (A + BK)^\top (\varepsilon P)(A + BK) > 0 \tag{7.27}$$

for all scalars $\varepsilon > 0$. Clearly, $\hat{P} + \varepsilon P > 0$ for all $\varepsilon > 0$. Moreover, by adding (7.27) and (7.24) we obtain

$$(A + BK)^\top (\hat{P} + \varepsilon P)(A + BK) - (\hat{P} + \varepsilon P) + Q + K^\top R K < 0$$

for all $\varepsilon > 0$. For $\varepsilon > 0$ sufficiently small, we also have that $x_0^\top (\hat{P} + \varepsilon P) x_0 < \gamma$. We conclude that for such ε, $\hat{P} + \varepsilon P > 0$ is a solution to (7.21) and (7.22). This proves the proposition. □

Next, we turn to the data-driven version of the suboptimal LQR problem. As in Section 7.1, we consider the model class $\mathcal{M}$ of all discrete-time linear input-state systems of the form (7.2) with given state space dimension n and input dimension m. Assume we have input-state data $\mathcal{D} := (U_-, X)$ on the time interval $[0, T]$ as given by (2.1). The set $\Sigma_{\mathcal{D}}$ of all systems in $\mathcal{M}$ that are consistent with the data is then equal to $\Sigma_{(U_-, X)}$ defined by (7.6). We assume that the data are generated by the true (but unknown) system $(A_{\text{true}}, B_{\text{true}})$, which is therefore assumed to be in $\Sigma_{(U_-,X)}$ itself. In the context of the suboptimal LQR problem the control objective $\mathcal{O}$ is: 'for the system with initial state x_0, interconnection with a state feedback controller yields a stable closed loop system and the associated cost is strictly less than γ'. With this in mind, we introduce the following notion of data informativity.

Definition 7.12. Let $x_0 \in \mathbb{R}^n$ and $\gamma > 0$. The data (U_-, X) are *informative for suboptimal linear quadratic regulation* if there exists a matrix K that is a suboptimal feedback gain for all $(A, B) \in \Sigma_{(U_-,X)}$.

We want to find conditions under which the data are informative for suboptimal linear quadratic regulation, and we want to obtain suboptimal controllers from data. These problems are stated more formally as follows.

Problem 7.13. Let $x_0 \in \mathbb{R}^n$ and $\gamma > 0$. Provide necessary and sufficient conditions under which the data (U_-, X) are informative for suboptimal linear quadratic regulation. Moreover, for data (U_-, X) that are informative, find a feedback gain K that is suboptimal for all $(A, B) \in \Sigma_{(U_-,X)}$.

An important ingredient in tackling this problem will be the observation that if the data (U_-, X) are informative for suboptimal linear quadratic regulation, they are necessarily informative for stabilization by state feedback. By Theorem 6.4, this holds if and only if there exists a right inverse $X_-^\sharp$ of X_- such that $X_+X_-^\sharp$ is stable. Moreover, K is a stabilizing feedback for all systems in $\Sigma_{(U_-,X)}$ if and only if $K = U_-X_-^\sharp$ for some $X_-^\sharp$ satisfying the above properties. This observation will be essential in proving the necessity parts in the following theorem.

Theorem 7.14. *Let $x_0 \in \mathbb{R}^n$ and $\gamma > 0$. The data (U_-, X) are informative for suboptimal linear quadratic regulation if and only if there exists a matrix $P > 0$ and a right inverse $X_-^\sharp$ of X_- such that*

$$(X_+X_-^\sharp)^\top P X_+X_-^\sharp - P + Q + (U_-X_-^\sharp)^\top R U_-X_-^\sharp < 0 \tag{7.28}$$

$$x_0^\top P x_0 < \gamma. \tag{7.29}$$

Moreover, K is a suboptimal feedback gain for all systems $(A, B) \in \Sigma_{(U_-,X)}$ if and only if it is of the form $K = U_-X_-^\sharp$ for some right inverse $X_-^\sharp$ satisfying (7.28) and (7.29).

Proof. To prove the 'if' parts of both statements, suppose that there exists a matrix $P > 0$ and a right inverse $X_-^\sharp$ such that (7.28) and (7.29) are satisfied. Define the controller $K := U_-X_-^\sharp$. For any $(A, B) \in \Sigma_{(U_-,X)}$ we have $X_+ = AX_- + BU_-$, which implies that $X_+X_-^\sharp = A + BK$. Substitution of the latter expression into (7.28) yields

$$(A + BK)^\top P(A + BK) - P + Q + K^\top RK < 0$$

which shows that there exists a K and $P > 0$ satisfying (7.21) and (7.22) for all $(A, B) \in \Sigma_{(U_-,X)}$. By Proposition 7.10, the data are informative for suboptimal LQR.

To prove the 'only if' parts of both statements, suppose that the data (U_-, X) are informative for suboptimal linear quadratic regulation. This means that there exists a feedback gain K and a matrix $P_{(A,B)} > 0$ such that

$$\begin{aligned} (A+BK)^\top P_{(A,B)}(A+BK) - P_{(A,B)} + Q + K^\top RK < 0 \\ x_0^\top P_{(A,B)} x_0 < \gamma \end{aligned}$$

for all $(A, B) \in \Sigma_{(U_-,X)}$. We emphasize that the matrix $P_{(A,B)}$ may depend on the particular system (A, B), but the feedback gain K is fixed by definition. Since K is such that $A+BK$ is stable for all $(A, B) \in \Sigma_{(U_-,X)}$, by the observation preceding the theorem statement we obtain that K is of the form $K = U_- X_-^\sharp$ for some right inverse $X_-^\sharp$ of X_-. This yields $A + BK = X_+ X_-^\sharp$. The matrix $A+BK$ is therefore the same for all $(A, B) \in \Sigma_{(U_-,X)}$. This implies the existence of a (common) $P > 0$ such that (7.28) and (7.29) are satisfied. $\square$

Note that the conditions of Theorem 7.14 are not ideal from a computational point of view since (7.28) depends nonlinearly on P and $X_-^\sharp$. Nonetheless, it is straightforward to reformulate these conditions in terms of linear matrix inequalities. In order to do so, let C and D be any choice of real matrices such that

$$\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} = \begin{bmatrix} C & D \end{bmatrix}^\top \begin{bmatrix} C & D \end{bmatrix}.$$

In addition, define $Z_- := CX_- + DU_-$. Then the following holds.

Corollary 7.15. *Let $x_0 \in \mathbb{R}^n$ and $\gamma > 0$. The data (U_-, X) are informative for suboptimal linear quadratic regulation if and only if there exist $Y \in \mathbb{S}^n$, $Y > 0$ and $\Theta \in \mathbb{R}^{T \times n}$ such that*

$$\begin{bmatrix} Y & \Theta^\top X_+^\top & \Theta^\top Z_-^\top \\ X_+\Theta & Y & 0 \\ Z_-\Theta & 0 & I \end{bmatrix} > 0 \tag{7.30}$$

$$\begin{bmatrix} \gamma & x_0^\top \\ x_0 & Y \end{bmatrix} > 0 \tag{7.31}$$

$$X_-\Theta = Y. \tag{7.32}$$

Moreover, K is a suboptimal feedback gain for all $(A, B) \in \Sigma_{(U_-,X)}$ if and only if $K = U_-\Theta Y^{-1}$ for some Y and Θ satisfying (7.30), (7.31) and (7.32).

Corollary 7.15 follows from Theorem 7.14 using standard manipulations. First a congruence transformation P^{-1} is applied to (7.28), after which a Schur complement argument and change of variables $Y := P^{-1}$ and $\Theta := X_-^\sharp Y$ yields (7.30), (7.31) and (7.32).

Remark 7.16. It is noteworthy that the conditions of Theorem 7.14 and Corollary 7.15 do not require that the data (U_-, X) contain enough information to uniquely identify the true system matrices $(A_{\text{true}}, B_{\text{true}})$. Clearly, Theorem 7.14 and Corollary 7.15 require the matrix X_- to have full row rank. This means that at least $T \geqslant n$ samples are needed to obtain a suboptimal controller from data. In comparison, recall from Theorem 3.1 that informativity for identification, i.e. the ability to uniquely recover $(A_{\text{true}}, B_{\text{true}})$ from the data, is equivalent to the full rank condition

$$\operatorname{rank} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = n + m$$

which requires the larger lower bound $T \geqslant n + m$ on the number of samples.

7.2.1 Illustrative example

As an illustration of the theory on the data-driven suboptimal LQR problem, in this subsection we will study controlled consensus dynamics of the form

$$x(t+1) = (I - 0.15L)\, x(t) + Bu(t) \tag{7.33}$$

where $x(t) \in \mathbb{R}^{20}$, $u(t) \in \mathbb{R}^{10}$, L is the Laplacian matrix of the graph G in Figure 7.1, and $B = \begin{bmatrix} I & 0 \end{bmatrix}^\top$, meaning that inputs are applied to the first 10 nodes. The goal of this example is to apply Corollary 7.15 to construct suboptimal controllers for the system (7.33) using data.

We choose the weight matrices as $Q = I$ and $R = I$, and define $x_0 \in \mathbb{R}^{20}$ entry-wise as $(x_0)_i = i$.

We start with a time horizon of $T = 20$ and collect data (U_-, X) where the entries of U_- and the initial state of the experiment $x(0)$ are drawn uniformly at random from $(0, 1)$. Given these data, we attempt to solve a semidefinite program (SDP) where the objective is to minimize γ subject to the constraints (7.30), (7.31) and (7.32). We use Yalmip, with Mosek as a solver. Next, we collect one additional sample of the input and state, and we solve the SDP again for the augmented data set. We continue this process up to a time horizon of $T = 30$.

We repeat this entire experiment for 100 trials and display the results in Figures 7.2 and 7.3. Figure 7.2 depicts the fraction of successful trials in which the constraints (7.30), (7.31) and (7.32) were feasible and a stabilizing controller was found. Note that a stabilizing controller was only found in 2 out of the 100

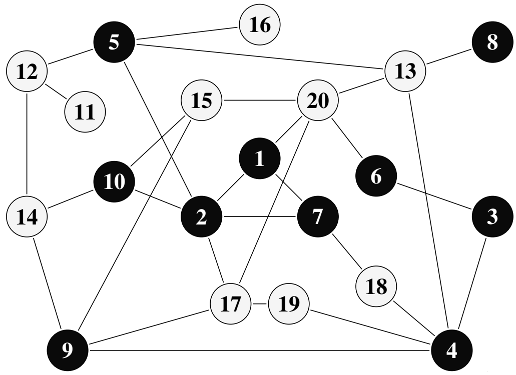


Figure 7.1: Graph G with leader vertices colored black.

trials for $T = 20$. This fraction rapidly increases to 0.88 for $T = 22$, while 100% of the trials were successful for $T \geqslant 24$. Figure 7.3 displays the minimum cost γ of the controller, averaged over all successful trials. The cost is very large for small sample size ($T = 20$) but decreases rapidly as the number of samples increases. Figure 7.3 therefore highlights an interesting trade-off between the sample size and the cost. Note that for $T = 30$, γ coincides with the optimal cost obtained from the (model-based) solution to the Riccati equation. This is as expected since $30 = n + m$ is the minimum number of samples from which the state and input matrices can be uniquely identified.

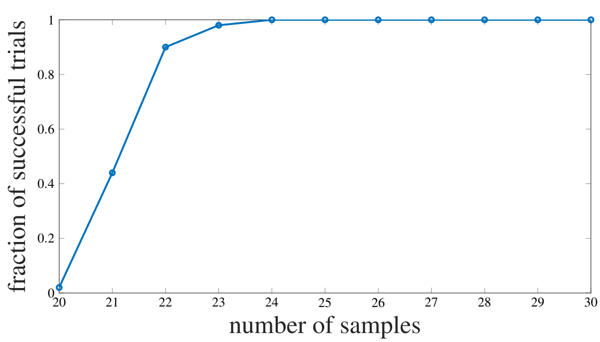


Figure 7.2: Fraction of successful trials as a function of T.

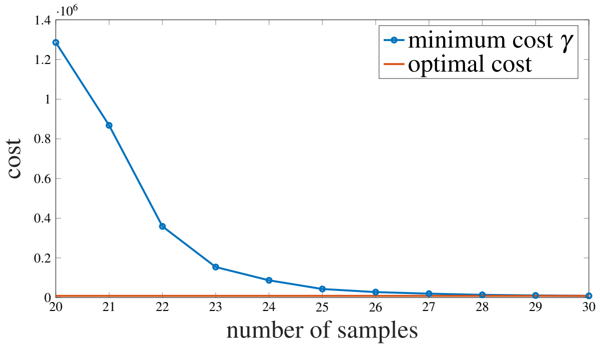


Figure 7.3: Average minimum cost as a function of T.

7.3 Notes and references

The basic material on the linear quadratic regulator problem provided in Section 7.1 is the discrete-time analogue of [160, Ch. 10]. See, in particular, [160, Thm. 10.18]. The solutions to the data-driven LQR problem in Theorems 7.5 and 7.8 are based on the paper [175]. The data-driven LQR problem was also solved using semidefinite programming in [44, Thm. 4], under the additional assumptions that the true system is controllable and the input sequence is persistently exciting of order $n + 1$. In this chapter, we have not assumed that the input data are persistently exciting. Instead, we have provided necessary and sufficient conditions under which the data are informative for linear quadratic regulation, i.e., conditions under which there exists a single feedback gain that is optimal for all data-consistent systems. Unlike in the case of stabilization (Section 6.1), these conditions basically imply that the data are informative for system identification.

In [44], the optimal feedback gain was found by minimizing the trace of a weighted sum of two matrix variables, subject to two LMI constraints. An attractive feature of the semidefinite program in Theorem 7.8 of this book is that the dimension of the unknown P is (only) $n \times n$. In comparison, the dimensions of the two unknowns in [44, Thm. 4] are $T \times n$ and $m \times m$, respectively. In general, the number of samples T is larger than n. In fact, in the case that the input is persistently exciting of order $n + 1$, we have $T \geqslant nm + n + m$. An additional feature of Theorem 7.8 is that P^+, i.e., the largest solution to the Riccati equation, is obtained from the data. This is useful since the optimal cost associated to any initial state x_0 can be computed as $x_0^\top P^+ x_0$.

The paper [63] also studies data-driven LQR, but using a different perspective. Indeed, in [63], the solution to the Riccati equation is approximated using a batch-form solution to the *Riccati difference equation*. A similar approach was used in the papers [1, 57, 152, 154] for the *finite horizon* data-driven LQR/LQG problem. In the setup of [63], the approximate solution to the Riccati equation is exact only if the number of data points tends to infinity. The main difference between our approach and the one in [63] is hence that the solution P^+ to the Riccati equation can be obtained exactly from *finite* data via Theorem 7.8.

The solution to the data-driven suboptimal LQR problem in Section 7.2 is based on the paper [176]. These results rely on an LMI characterization under which a given feedback gain is suboptimal, see Proposition 7.10. In order to prove this proposition, we have used the auxiliary Lemma 7.11 on the uniqueness of solutions to (discrete-time) Lyapunov equations. A proof of this lemma can be found in [12], see Theorem 4.50.

8

Data-driven $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control design

In this chapter we will study two main design problems, namely the data-driven $\mathcal{H}_2$ suboptimal control design problem and the data-driven $\mathcal{H}_\infty$ control problem.

Two versions of data-driven $\mathcal{H}_2$ suboptimal control design will be considered. In the first of these, to be treated in Section 8.1, measurements of the external disturbance input will be part of the data. In the second version, in Section 8.2, the disturbance input will be assumed to be unknown, and we will deal with noisy data.

The second main subject of this chapter is the data-driven $\mathcal{H}_\infty$ control problem. We define the property of informativity for $\mathcal{H}_\infty$ control for noisy input-state data, and establish necessary and sufficient conditions on the data to be informative. These conditions are formulated in terms of feasibility of certain LMIs involving the data matrices and the desired performance. Solutions to these LMIs also lead to $\mathcal{H}_\infty$ suboptimal state feedback control laws.

8.1 $\mathcal{H}_2$ suboptimal control design with disturbance data

In this section we will study the data-driven $\mathcal{H}_2$ suboptimal control problem. Before embarking on data-driven design, we will first review some basic material on $\mathcal{H}_2$ suboptimal control. In that context, we consider a given system

$$x(t+1) = Ax(t) + Bu(t) + Ew(t) \tag{8.1a}$$
$$z(t) = Cx(t) + Du(t) \tag{8.1b}$$

where x is the n-dimensional state, u the m-dimensional control input, w a d-dimensional disturbance input (in the sequel also referred to as noise input) and z the p-dimensional performance output. The real matrices A, B, C, D and E are of appropriate dimensions. A given feedback law $u = Kx$ yields the closed-loop system

$$x(t+1) = (A + BK)x(t) + Ew(t) \tag{8.2a}$$
$$z(t) = (C + DK)x(t). \tag{8.2b}$$

Associated with (8.2), we consider the cost functional

$$J_{\mathcal{H}_2}(K) := \sum_{t=0}^{\infty} \operatorname{tr}\left(T_K^\top(t) T_K(t)\right)$$

where $T_K(t) := (C+DK)(A+BK)^t E$ is the closed-loop impulse response matrix from w to z and, as before, $\operatorname{tr}(\cdot)$ denotes trace. The cost $J_{\mathcal{H}_2}(K)$ is called the $\mathcal{H}_2$ cost of the feedback gain K since it is equal to the squared $\mathcal{H}_2$ norm of the transfer function from w to z in (8.2). It is well-known that the $\mathcal{H}_2$ cost of a given stabilizing K can be computed using the observability Gramian. Indeed for a stabilizing K, the unique solution P to the Lyapunov equation

$$(A+BK)^\top P(A+BK) - P + (C+DK)^\top (C+DK) = 0 \tag{8.3}$$

is related to the $\mathcal{H}_2$ cost by $\operatorname{tr}(E^\top P E) = J_{\mathcal{H}_2}(K)$. For a given tolerance $\gamma > 0$, the $\mathcal{H}_2$ suboptimal control problem is now the problem of finding a gain K (if it exists) such that $A + BK$ is stable and $J_{\mathcal{H}_2}(K) < \gamma$. Such a K is called an $\mathcal{H}_2$ *suboptimal feedback gain.* Similar to Proposition 7.10 the following proposition gives conditions under which a given K is an $\mathcal{H}_2$ suboptimal feedback gain.

Proposition 8.1. *Let $\gamma > 0$. The matrix K is an $\mathcal{H}_2$ suboptimal feedback gain if and only if there exists a matrix $P > 0$ such that*

$$\begin{aligned} (A+BK)^\top P(A+BK) - P + (C+DK)^\top (C+DK) &< 0 \\ \operatorname{tr}\left(E^\top P E\right) &< \gamma. \end{aligned}$$

Clearly, the suboptimal LQR problem can be viewed as a special case of the $\mathcal{H}_2$ suboptimal control problem. Indeed, the $\mathcal{H}_2$ problem boils down to the LQR problem if $E = x_0$, $C^\top C = Q$, $D^\top D = R$ and $C^\top D = 0$. However, as we will see in the sequel, the data-driven versions of these problems are different in the way that data are collected.

With the basic material now available, we turn our attention to the data-driven version of the $\mathcal{H}_2$ suboptimal control problem. For this, consider the true (but unknown) system

$$x(t+1) = A_{\text{true}} x(t) + B_{\text{true}} u(t) + E_{\text{true}} w(t) \tag{8.4}$$

where the system matrices A_{true}, B_{true} and E_{true} are unknown real matrices of given (known) dimensions. In order to quantify the performance of the system, we introduce a performance output

$$z(t) = Cx(t) + Du(t)$$

where the matrices C and D are known real matrices. We embed the unknown system (8.4) into the model class $\mathcal{M}$ of all discrete-time linear systems (with given state space dimension n, control input dimension m and disturbance input dimension d) of the form

$$x(t+1) = Ax(t) + Bu(t) + Ew(t). \tag{8.5}$$

Suppose we collect data on the time interval $[0, T]$. This leads to input-state data (U_-, X). We also collect corresponding measurements of the disturbance leading to disturbance data

$$W_- := W_{[0,T-1]} = \begin{bmatrix} w(0) & w(1) & \cdots & w(T-1) \end{bmatrix}.$$

Note that we assume that samples W_- of the disturbance inputs are available as part of the data.

In this setup, all systems (A, B, E) in the model class $\mathcal{M}$ that are consistent with the data (U_-, W_-, X) are given by

$$\Sigma_{(U_-,W_-,X)} := \left\{ (A, B, E) \mid X_+ = \begin{bmatrix} A & B & E \end{bmatrix} \begin{bmatrix} X_- \\ U_- \\ W_- \end{bmatrix} \right\}.$$

We can now state the following notion of data informativity for $\mathcal{H}_2$ suboptimal control.

Definition 8.2. Let $\gamma > 0$. The data (U_-, W_-, X) are *informative for $\mathcal{H}_2$ suboptimal control* if there exists a K that is an $\mathcal{H}_2$ suboptimal feedback gain for all $(A, B, E) \in \Sigma_{(U_-,W_-,X)}$.

As before, we are interested in both data informativity conditions and a control design procedure. We formalize this in the following problem.

Problem 8.3. Let $\gamma > 0$. Provide necessary and sufficient conditions under which the data (U_-, W_-, X) are informative for $\mathcal{H}_2$ suboptimal control. Moreover, for data (U_-, W_-, X) that are informative, find a feedback gain K that is $\mathcal{H}_2$ suboptimal for all $(A, B, E) \in \Sigma_{(U_-,W_-,X)}$.

In the remainder of this section we will resolve the data-driven $\mathcal{H}_2$ suboptimal control problem as formulated in Problem 8.3. In order to do so, as a first step we need to extend Theorem 6.4 to systems with disturbances. We call the data (U_-, W_-, X) *informative for stabilization by state feedback* if there exists K such that $A + BK$ is stable for all $(A, B, E) \in \Sigma_{(U_-,W_-,X)}$.

Lemma 8.4. *The data (U_-, W_-, X) are informative for stabilization by state feedback if and only if there exists a right inverse $X_-^\sharp$ of X_- with the properties that $X_+X_-^\sharp$ is stable and $W_-X_-^\sharp = 0$.*

Moreover, K is a stabilizing controller for all systems in $\Sigma_{(U_-,W_-,X)}$ if and only if $K = U_- X_-^\sharp$, where $X_-^\sharp$ satisfies the above properties.

Proof. The proof follows a similar line as that of Theorem 6.4. To prove the 'if' part of both statements, suppose that there exists a right inverse $X_-^\sharp$ such that $X_+X_-^\sharp$ is stable and $W_-X_-^\sharp = 0$. Define $K := U_-X_-^\sharp$. Then $X_+X_-^\sharp = A + BK$ for all $(A, B, E) \in \Sigma_{(U_-,W_-,X)}$. Hence $A + BK$ is stable for all $(A, B, E) \in \Sigma_{(U_-,W_-,X)}$ and $K = U_-X_-^\sharp$ is stabilizing.

To prove the 'only if' parts, suppose that the data are informative for stabilization by state feedback. Let K be stabilizing for all systems in $\Sigma_{(U_-,W_-,X)}$. Define the subspace

$$\Sigma^0_{(U_-,W_-,X)} := \left\{ (A_0, B_0, E_0) \mid 0 = \begin{bmatrix} A_0 & B_0 & E_0 \end{bmatrix} \begin{bmatrix} X_- \\ U_- \\ W_- \end{bmatrix} \right\}.$$

The matrix $A+BK+\alpha(A_0+B_0K)$ is stable for all $\alpha \in \mathbb{R}$ and all $(A_0, B_0, E_0) \in \Sigma^0_{(U_-,W_-,X)}$. Thus we have

$$\rho\left(\frac{1}{\alpha}(A + BK) + A_0 + B_0K\right) \leqslant \frac{1}{\alpha} \quad \forall\, \alpha \geqslant 1$$

where $\rho(\cdot)$ denotes spectral radius. We take the limit as $\alpha \to \infty$, and conclude by continuity of the spectral radius that A_0+B_0K is nilpotent for all $(A_0, B_0, E_0) \in \Sigma^0_{(U_-,W_-,X)}$. Note that $(A_0, B_0, E_0) \in \Sigma^0_{(U_-,W_-,X)}$ implies that

$$\left((A_0 + B_0K)^\top A_0, (A_0 + B_0K)^\top B_0, (A_0 + B_0K)^\top E_0\right)$$

is also a member of $\Sigma^0_{(U_-,W_-,X)}$. This implies that the matrix $(A_0+B_0K)^\top(A_0+B_0K)$ is nilpotent for all (A_0, B_0, E_0). The only symmetric nilpotent matrix is zero, thus $A_0 + B_0K = 0$ for all $(A_0, B_0, E_0) \in \Sigma^0_{(U_-,W_-,X)}$. We conclude that

$$\ker \begin{bmatrix} X_-^\top & U_-^\top & W_-^\top \end{bmatrix} \subseteq \ker \begin{bmatrix} I & K^\top & 0 \end{bmatrix}$$

equivalently,

$$\operatorname{im} \begin{bmatrix} I \\ K \\ 0 \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} X_- \\ U_- \\ W_- \end{bmatrix}.$$

This means that there exists a right inverse $X_-^\sharp$ of X_- such that $K = U_-X_-^\sharp$ and $W_-X_-^\sharp = 0$. Clearly, $X_+X_-^\sharp = A + BK$ for all $(A, B, E) \in \Sigma_{(U_-,W_-,X)}$, hence $X_+X_-^\sharp$ is stable. $\square$

The following theorem provides necessary and sufficient conditions for data informativity for the suboptimal $\mathcal{H}_2$ problem. It also characterizes all suboptimal controllers in terms of the data. Again define $Z_- := CX_- + DU_-$.

Theorem 8.5. *Let $\gamma > 0$. The data (U_-, W_-, X) are informative for $\mathcal{H}_2$ suboptimal control if and only if at least one of the following two conditions is satisfied:*

(a) *There exists a right inverse $X_-^\sharp$ such that $X_+X_-^\sharp$ is stable and*

$$\begin{bmatrix} W_- \\ Z_- \end{bmatrix} X_-^\sharp = 0. \tag{8.6}$$

(b) *There exist right inverses $X_-^\sharp$ and $W_-^\sharp$ such that $X_+X_-^\sharp$ is stable, $W_-X_-^\sharp = 0$,*

$$\begin{bmatrix} X_- \\ U_- \end{bmatrix} W_-^\sharp = 0 \tag{8.7}$$

and the unique solution P to

$$(X_-^\sharp)^\top \left(X_+^\top P X_+ - X_-^\top P X_- + Z_-^\top Z_-\right) X_-^\sharp = 0 \tag{8.8}$$

has the property that

$$\operatorname{tr}\left((X_+W_-^\sharp)^\top P X_+ W_-^\sharp\right) < \gamma. \tag{8.9}$$

Moreover, K is an $\mathcal{H}_2$ suboptimal controller for all $(A, B, E) \in \Sigma_{(U_-,W_-,X)}$ if and only if $K = U_-X_-^\sharp$, where $X_-^\sharp$ satisfies the conditions of (a) or (b).

Remark 8.6. The interpretation of Theorem 8.5 is as follows. Note that both condition (a) and (b) require the existence of $X_-^\sharp$ such that $X_+X_-^\sharp$ is stable and $W_-X_-^\sharp = 0$. These conditions are necessary for the existence of a stabilizing controller by Lemma 8.4. In condition (a) it is further required that $X_-^\sharp$ satisfies $Z_-X_-^\sharp = 0$, which means that the output of all systems in $\Sigma_{(U_-,W_-,X)}$ can be made identically equal to zero (hence their $\mathcal{H}_2$ norm is zero). In condition (b), the properties of $W_-^\sharp$ imply that $E_{\text{true}} = X_+W_-^\sharp$ can be uniquely identified from the data. Similar to the suboptimal LQR problem, it is generally not required that A_{true} and B_{true} can be uniquely identified from the data.

Proof. We first prove the 'if' parts of both statements. Suppose that condition (a) is satisfied and let $K := U_-X_-^\sharp$. By Lemma 8.4, $A + BK$ is stable for all

$(A, B, E) \in \Sigma_{(U_-,W_-,X)}$. As $Z_- X_-^\sharp = 0$ we have $C + DU_- X_-^\sharp = C + DK = 0$. This means that the $\mathcal{H}_2$ cost of (8.2) is zero for all $(A, B, E) \in \Sigma_{(U_-,W_-,X)}$. We conclude that the data are informative for $\mathcal{H}_2$ suboptimal control and K is an $\mathcal{H}_2$ suboptimal controller.

Next suppose that condition (b) is satisfied, and let $K := U_- X_-^\sharp$ where $X_-^\sharp$ satisfies the conditions of (b). Clearly, $A + BK = X_+ X_-^\sharp$ is stable for all $(A, B, E) \in \Sigma_{(U_-,W_-,X)}$. By the properties of $W_-^\sharp$, $(A, B, E) \in \Sigma_{(U_-,W_-,X)}$ implies $E = E_{\text{true}} = X_+ W_-^\sharp$. In view of (8.8) and (8.9) we see that for any $(A, B, E_{\text{true}}) \in \Sigma_{(U_-,W_-,X)}$ the unique solution P to (8.3) satisfies

$$\operatorname{tr}(E_{\text{true}}^\top P E_{\text{true}}) < \gamma.$$

Therefore, the data are informative for $\mathcal{H}_2$ suboptimal control and K is $\mathcal{H}_2$ suboptimal.

Subsequently, we prove the 'only if' parts of both statements. Suppose that the data are informative for $\mathcal{H}_2$ suboptimal control and let K be an $\mathcal{H}_2$ suboptimal controller for all $(A, B, E) \in \Sigma_{(U_-,W_-,X)}$. By Lemma 8.4, there exists a right inverse $X_-^\sharp$ such that $X_+ X_-^\sharp$ is stable and $W_- X_-^\sharp = 0$. Also, the feedback K is of the form $K = U_- X_-^\sharp$ and $A + BK = X_+ X_-^\sharp$ for all $(A, B, E) \in \Sigma_{(U_-,W_-,X)}$. By stability of $X_+ X_-^\sharp$ the Lyapunov equation (8.8) has a unique solution $P \geqslant 0$. The matrix P satisfies $\operatorname{tr}(E^\top P E) < \gamma$ for all $(A, B, E) \in \Sigma_{(U_-,W_-,X)}$. Therefore, we have

$$\operatorname{tr}\left((E + \alpha E_0)^\top P (E + \alpha E_0)\right) < \gamma \tag{8.10}$$

for all $(A, B, E) \in \Sigma_{(U_-,W_-,X)}$, $(A_0, B_0, E_0) \in \Sigma^0_{(U_-,W_-,X)}$ and $\alpha \in \mathbb{R}$. We divide both sides of (8.10) by α^2 and take the limit as $\alpha \to \infty$. Then, by continuity of the trace we obtain $\operatorname{tr}(E_0^\top P E_0) = 0$, which yields $P E_0 = 0$ for all $(A_0, B_0, E_0) \in \Sigma^0_{(U_-,W_-,X)}$. We claim that this implies that either $P = 0$ or $E_0 = 0$ for all $(A_0, B_0, E_0) \in \Sigma^0_{(U_-,W_-,X)}$.

Indeed, suppose that this claim is not true. Then $P \neq 0$ and there exists a triple $(A_0, B_0, E_0) \in \Sigma^0_{(U_-,W_-,X)}$ such that $E_0 \neq 0$. Then, clearly, there exists $f, g \in \mathbb{R}^n$ such that $Pf \neq 0$ and $g^\top E_0 \neq 0$, so $Pfg^\top E_0 \neq 0$. Note that for any $F \in \mathbb{R}^{n \times n}$ we have $(FA_0, FB_0, FE_0) \in \Sigma^0_{(U_-,W_-,X)}$. On the other hand, for $F := fg^\top$ we have $PFE_0 \neq 0$. This is a contradiction, which proves our claim.

Now, in the case that $P = 0$ we obtain $Z_- X_-^\sharp = 0$ so condition (a) is satisfied. In the case that $E_0 = 0$ for all $(A_0, B_0, E_0) \in \Sigma^0_{(U_-,W_-,X)}$ we obtain

$$\ker \begin{bmatrix} X_-^\top & U_-^\top & W_-^\top \end{bmatrix} \subseteq \ker \begin{bmatrix} 0 & 0 & I \end{bmatrix}$$

equivalently,

$$\operatorname{im} \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} X_- \\ U_- \\ W_- \end{bmatrix}.$$

As a consequence, there exists a right inverse $W_-^\sharp$ such that $X_- W_-^\sharp = 0$ and $U_- W_-^\sharp = 0$. This means that $(A, B, E) \in \Sigma_{(U_-, W_-, X)}$ implies $E = E_{\text{true}} = X_+ W_-^\sharp$. Hence (8.9), and therefore (b), holds. In both cases, the controller K is of the form $K = U_- X_-^\sharp$, where $X_-^\sharp$ satisfies either (a) or (b). □

Similar to Corollary 7.15 we can reformulate Theorem 8.5 in terms of linear matrix inequalities.

Corollary 8.7. *Let $\gamma > 0$. The data (U_-, W_-, X) are informative for $\mathcal{H}_2$ suboptimal control if and only if at least one of the following two conditions is satisfied:*

(a) *There exists a $\Theta \in \mathbb{R}^{T \times n}$ such that $X_- \Theta = (X_- \Theta)^\top$,*

$$\begin{bmatrix} W_- \\ Z_- \end{bmatrix} \Theta = 0 \text{ and } \begin{bmatrix} X_- \Theta & \Theta^\top X_+^\top \\ X_+ \Theta & X_- \Theta \end{bmatrix} > 0. \tag{8.11}$$

(b) *There exist a right inverse $W_-^\sharp$ and matrices $Y \in \mathbb{S}^n$, $Y > 0$ and $\Theta \in \mathbb{R}^{T \times n}$ such that $X_- \Theta$ is symmetric, the matrices $W_- \Theta$, $X_- W_-^\sharp$ and $U_- W_-^\sharp$ are zero, and*

$$\begin{bmatrix} X_- \Theta & \Theta^\top X_+^\top & \Theta^\top Z_-^\top \\ X_+ \Theta & X_- \Theta & 0 \\ Z_- \Theta & 0 & I \end{bmatrix} > 0 \tag{8.12}$$

$$\begin{bmatrix} Y & (W_-^\sharp)^\top X_+^\top \\ X_+ W_-^\sharp & X_- \Theta \end{bmatrix} > 0 \tag{8.13}$$

$$\operatorname{tr}(Y) < \gamma. \tag{8.14}$$

Moreover, K is an $\mathcal{H}_2$ suboptimal controller for all $(A, B, E) \in \Sigma_{(U_-, W_-, X)}$ if and only if $K = U_- \Theta (X_- \Theta)^{-1}$, where Θ satisfies the conditions of (a) or (b).

Proof. We first prove the 'if' part. We will first show that condition (a) implies condition (a) of Theorem 8.5. Let Θ be such that $X_- \Theta$ is symmetric and (8.11) holds. Then $X_- \Theta > 0$ and $X_-^\sharp := \Theta (X_- \Theta)^{-1}$ is a right-inverse of X_-. Also, (8.6) holds. Now define $Q := (X_- \Theta)^{-1}$. By taking a suitable Schur complement we then obtain $Q - (X_+X_-^\sharp)^\top Q(X_+X_-^\sharp) > 0$, from which it follows that $X_+X_-^\sharp$ is stable.

Next, we will prove that condition (b) implies condition (b) of Theorem 8.5. By (8.12), again $X_-\Theta > 0$, $X_-^\sharp := \Theta(X_-\Theta)^{-1}$ is a right-inverse of X_- and $X_+X_-^\sharp$ is stable. Also $W_-X_-^\sharp = 0$ and (8.7) holds. Applying standard Schur complement arguments to (8.12), it is the easily seen that $Q = (X_-\Theta)^{-1}$ satisfies the strict Lyapunov inequality

$$(X_-^\sharp)^\top \left(X_+^\top Q X_+ - X_-^\top Q X_- + Z_-^\top Z_-\right) X_-^\sharp < 0.$$

Now let $P \geqslant 0$ be the unique solution to the Lyapunov equation (8.8). Then the difference $\Delta := Q - P$ satisfies

$$(X_-^\sharp)^\top \left(X_+^\top \Delta X_+ - X_-^\top \Delta X_-\right) X_-^\sharp < 0,$$

which implies $\Delta > 0$ so $P < Q$. Using (8.13) we find

$$Y - (X_+W_-^\sharp)^\top Q(X_+W_-^\sharp) > 0$$

which leads to

$$(X_+W_-^\sharp)^\top P(X_+W_-^\sharp) < Y.$$

By applying (8.14) this then yields (8.9).

We now turn to the proof of the 'only if' part. We will show that condition (a) of Theorem 8.5 implies condition (a) of Corollary 8.7. Let X_- be a right inverse of X_- such that $X_+X_-^\sharp$ is stable. The stability of $X_+X_-^\sharp$ implies the existence of a $Q > 0$ such that

$$Q - (X_+X_-^\sharp)^\top Q(X_+X_-^\sharp) > 0.$$

Next, define $\Theta := X_-^\sharp Q^{-1}$. Then $Q = (X_-\Theta)^{-1}$ and it is easily verified that

$$X_-\Theta - (X_+\Theta)^\top (X_-\Theta)^{-1}(X_+\Theta) > 0.$$

Using the Schur complement we conclude that (8.11) holds.

Next, we show that condition (b) of Theorem 8.5 implies condition (b) of Corollary 8.7. For any $\varepsilon > 0$, let $Q_\varepsilon > 0$ be the unique solution of the Lyapunov equation

$$(X_-^\sharp)^\top \left(X_+^\top Q_\varepsilon X_+ - X_-^\top Q_\varepsilon X_- + Z_-^\top Z_-\right) X_-^\sharp + \varepsilon I = 0.$$

Clearly, Q_ε converges to the solution P of (8.8) as $\varepsilon \downarrow 0$. Using this fact together with (8.9), pick an $\varepsilon > 0$ sufficiently small such that

$$\operatorname{tr}\left((X_+W_-^\sharp)^\top Q_\varepsilon X_+W_-^\sharp\right) < \gamma.$$

In the sequel, denote this Q_ε simply by Q. Note that Q satisfies

$$(X_-^\sharp)^\top \left(X_+^\top Q X_+ - X_-^\top Q X_- + Z_-^\top Z_-\right) X_-^\sharp < 0.$$

Define $\Theta := X_-^\sharp Q^{-1}$. Using Schur complements it is then easily seen that the inequality (8.12) holds. It remains to be proven that Y exists such that (8.13) and (8.14) hold. Note that $\operatorname{tr}((X_+ W_-^\sharp)^\top (X_-\Theta)^{-1} X_+ W_-^\sharp) < \gamma$. Choose any $Y > (X_+ W_-^\sharp)^\top (X_-\Theta)^{-1} X_+ W_-^\sharp$ such that $\operatorname{tr}(Y) < \gamma$.

Finally, the statement about the the feedback gain K is an immediate consequence of the previous. This completes the proof of Corollary 8.7. □

8.2 $\mathcal{H}_2$ suboptimal control design without disturbance data

In the setup considered in the previous section, part of our data consisted of samples of the noise input acting on the true, unknown system. Thus, our data set $\mathcal{D}$ consisted of data on the control input, the state, and the noise input. In the current section we turn our attention to the situation that the noise input is unknown, and only input-state data are available to obtain $\mathcal{H}_2$ suboptimal controllers.

Consider the true (but unknown) system

$$x(t+1) = A_{\text{true}} x(t) + B_{\text{true}} u(t) + w(t) \tag{8.15}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input and $w(t) \in \mathbb{R}^n$ is an unknown noise input. The matrices A_{true} and B_{true} denote the unknown state and input matrices. Different from the setup in Section 8.1, we assume that the noise input matrix E is known, and, in fact, equal to the $n \times n$ identity matrix.

We embed this unknown system into the model class $\mathcal{M}$ of all input-state systems with unknown noise inputs, of fixed dimensions n and m, of the form

$$x(t+1) = Ax(t) + Bu(t) + w(t). \tag{8.16}$$

Suppose that we have data on the time interval $[0, T]$. These data are given by (U_-, X). In contrast with the setup in Section 8.1, the noise input w is assumed to be unknown. In particular this means that $w(0), w(1), \dots, w(T-1)$ are not measured, and are therefore not part of the data.

Using the noise model that was introduced in Section 3.4, we will however assume that the matrix $W_- = W_{[0,T-1]}$ of noise samples satisfies the quadratic matrix inequality (3.18) for a given, known, matrix $\Phi \in \mathbf{\Pi}_{n,T}$. The set $\Sigma_{\mathcal{D}}$ of all systems in $\mathcal{M}$ that are consistent with the data (U_-, X) is equal to the set of all systems (A, B) satisfying

$$X_+ = AX_- + BU_- + W_- \tag{8.17}$$

for some W_- satisfying the quadratic matrix inequality (3.18). In other words,

$$\Sigma_{\mathcal{D}} = \{(A, B) \mid (8.17) \text{ holds for some } W_- \text{ satisfying } (3.18)\}. \tag{8.18}$$

Recall from Lemma 3.16 that, in fact,

$$\Sigma_{\mathcal{D}} = \{(A, B) \mid (3.27) \text{ is satisfied}\}$$

In order to quantify performance, we associate to (8.15) a performance output

$$z(t) = Cx(t) + Du(t) \tag{8.19}$$

where $z(t) \in \mathbb{R}^p$, and C and D are known matrices. For any $(A, B) \in \Sigma_{\mathcal{D}}$, the feedback law $u = Kx$ yields the closed-loop system

$$\begin{aligned} x(t+1) &= (A + BK)x(t) + w(t) \\ z(t) &= (C + DK)x(t). \end{aligned} \tag{8.20}$$

Associated with (8.20), we again consider the $\mathcal{H}_2$ cost functional

$$J_{\mathcal{H}_2}(K) := \sum_{t=0}^{\infty} \operatorname{tr}\left(T_K^\top(t) T_K(t)\right),$$

where $T_K(t) := (C + DK)(A + BK)^t$ is the closed-loop impulse response matrix from w to z. Let $\gamma > 0$. We have that $A + BK$ is stable and $J_{\mathcal{H}_2}(K) < \gamma$ if and only if there exists a matrix $P > 0$ such that

$$\begin{aligned} (A + BK)^\top P(A + BK) - P + (C + DK)^\top (C + DK) < 0 \\ \operatorname{tr}(P) < \gamma. \end{aligned} \tag{8.21}$$

The data-driven $\mathcal{H}_2$ suboptimal control problem problem entails the computation of a feedback gain K from data such that $J_{\mathcal{H}_2}(K) < \gamma$ for all (A, B) that are consistent with the data. Similar to the setup for quadratic stabilization in Section 6.3, we restrict attention to a matrix P that is common for all (A, B). This leads to the following natural definition.

Definition 8.8. Let $\gamma > 0$. The data (U_-, X) are *informative for $\mathcal{H}_2$ suboptimal control* if there exist matrices $P > 0$ and K such that (8.21) holds for all $(A, B) \in \Sigma_{\mathcal{D}}$.

With the theory of Section A.3 available, characterizing informativity for $\mathcal{H}_2$ suboptimal control essentially boils down to massaging the inequalities (8.21) such that they are amenable to design. To this end, note that the first inequality of (8.21) is equivalent to

$$Y - A_{Y,L}^\top P A_{Y,L} - C_{Y,L}^\top C_{Y,L} > 0$$

where we defined $A_{Y,L} := AY + BL$ and $C_{Y,L} := CY + DL$ with $Y := P^{-1}$ and $L := KY$. Using a Schur complement argument, this is equivalent to

$$\begin{bmatrix} Y - C_{Y,L}^\top C_{Y,L} & A_{Y,L}^\top \\ A_{Y,L} & Y \end{bmatrix} > 0. \tag{8.22}$$

Now, (8.22) holds if and only if

$$Y - C_{Y,L}^\top C_{Y,L} > 0 \tag{8.23}$$
$$Y - A_{Y,L}(Y - C_{Y,L}^\top C_{Y,L})^{-1} A_{Y,L}^\top > 0. \tag{8.24}$$

Note that (8.23) is independent of A and B. In turn, we can write (8.24) as

$$\begin{bmatrix} I \\ \hdashline A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} Y & 0 \\ \hdashline 0 & -\begin{bmatrix} Y \\ L \end{bmatrix} (Y - C_{Y,L}^\top C_{Y,L})^{-1} \begin{bmatrix} Y \\ L \end{bmatrix}^\top \end{bmatrix} \begin{bmatrix} I \\ \hdashline A^\top \\ B^\top \end{bmatrix} > 0. \tag{8.25}$$

The inequality (8.25) is of a form where A and B appear on the left and their transposes appear on the right, analogous to (6.24). As in Chapter 6, let

$$N = \begin{bmatrix} N_{11} & N_{12} \\ \hdashline N_{12}^\top & N_{22} \end{bmatrix} := \begin{bmatrix} I & X_+ \\ \hdashline 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ \hdashline 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top \tag{8.26}$$

and let M be defined by

$$M = \begin{bmatrix} M_{11} & M_{12} \\ \hdashline M_{12}^\top & M_{22} \end{bmatrix} := \begin{bmatrix} Y & 0 \\ \hdashline 0 & -\begin{bmatrix} Y \\ L \end{bmatrix} (Y - C_{Y,L}^\top C_{Y,L})^{-1} \begin{bmatrix} Y \\ L \end{bmatrix}^\top \end{bmatrix}. \tag{8.27}$$

Then, for given $\gamma > 0$, informativity for $\mathcal{H}_2$ suboptimal control quadratic stability holds if and only if there exist matrices $Y > 0$ and L such that $\operatorname{tr}(Y^{-1}) < \gamma$, $Y - C_{Y,L}^\top C_{Y,L} > 0$ and

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} > 0 \quad \text{for all } Z \in \mathbb{R}^{(n+m)\times n} \text{ such that } \begin{bmatrix} I \\ Z \end{bmatrix}^\top N \begin{bmatrix} I \\ Z \end{bmatrix} \geqslant 0 \tag{8.28}$$

with Z given by

$$Z := \begin{bmatrix} A^\top \\ B^\top \end{bmatrix}.$$

Using the sets (A.3) and (A.12) (see Section A.2), condition (8.28) can be equivalently formulated as

$$\mathcal{Z}_{n+m}(N) \subseteq \mathcal{Z}_{n+m}^+(M). \tag{8.29}$$

As such, we are in a position to apply Corollary A.23. In fact, we derive the following theorem.

Theorem 8.9. *Let $\gamma > 0$. Then the data (U_-, X) are informative for $\mathcal{H}_2$ suboptimal control if and only if there exist matrices $Y \in \mathbb{S}^n$, $Y > 0$, $Z \in \mathbb{S}^n$ and $L \in \mathbb{R}^{m\times n}$, and scalars $\alpha \geqslant 0$ and $\beta > 0$ satisfying*

$$\begin{bmatrix} Y-\beta I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y & 0 \\ 0 & 0 & 0 & L & 0 \\ 0 & Y & L^\top & Y & C_{Y,L}^\top \\ 0 & 0 & 0 & C_{Y,L} & I \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^\top \geqslant 0$$
$$\begin{bmatrix} Y & C_{Y,L}^\top \\ C_{Y,L} & I \end{bmatrix} > 0, \quad \begin{bmatrix} Z & I \\ I & Y \end{bmatrix} \geqslant 0, \quad \operatorname{tr} Z < \gamma. \tag{8.30}$$

Moreover, if Y and L satisfy (8.30) then $K := LY^{-1}$ is such that $A + BK$ is stable and $J_{\mathcal{H}_2}(K) < \gamma$ for all $(A, B) \in \Sigma_{\mathcal{D}}$.

Proof. Suppose that (8.30) is feasible and define $P := Y^{-1}$ and $K := LP$. The last two inequalities of (8.30) imply that $\operatorname{tr}(P) < \gamma$. We now compute the Schur complement of the first LMI in (8.30) with respect to the diagonal block

$$\begin{bmatrix} Y & C_{Y,L}^\top \\ C_{Y,L} & I \end{bmatrix}.$$

We thereby make use of the fact that this block is nonsingular by the second LMI of (8.30). The computation of the Schur complement results in

$$M - \alpha N \geqslant \begin{bmatrix} \beta I & 0 \\ 0 & 0 \end{bmatrix} \tag{8.31}$$

where M is defined in (8.27) and N is defined in (8.26). We thus conclude that (8.28) is satisfied. As such, (8.24) holds for all $(A, B) \in \Sigma_{\mathcal{D}}$. Note that (8.23) holds by the second LMI of (8.30). Therefore, we conclude that (8.21) holds for all $(A, B) \in \Sigma_{\mathcal{D}}$. In other words, the data (U_-, X) are informative for $\mathcal{H}_2$ suboptimal control, and $K = LY^{-1}$ is a suitable controller.

Conversely, suppose that the data (U_-, X) are informative for $\mathcal{H}_2$ suboptimal control with performance bound γ. Then there exist matrices $P > 0$ and K such that (8.21) holds for all $(A, B) \in \Sigma_{\mathcal{D}}$. Define $Y := P^{-1}$, $L := KY$ and $Z := P$. Clearly, the last two inequalities of (8.30) are satisfied by definition of Z. In addition, we know that (8.23) and (8.24) hold for all $(A, B) \in \Sigma_{\mathcal{D}}$. By (8.23), the second LMI of (8.30) is satisfied. To prove that the first LMI of (8.30) also holds, we want to apply Corollary A.23. Note that we have already verified the assumptions of this theorem for the matrix N in (8.26), see the discussion preceding Theorem 3.17. In addition, we note that

$$M_{22} = -\begin{bmatrix} Y \\ L \end{bmatrix} (Y - C_{Y,L}^\top C_{Y,L})^{-1} \begin{bmatrix} Y \\ L \end{bmatrix}^\top \leqslant 0$$

since $Y - C_{Y,L}^\top C_{Y,L} > 0$. Hence, Corollary A.23 is applicable. We conclude that there exist $\alpha \geqslant 0$ and $\beta > 0$ such that (8.31) holds. Using a Schur complement argument, we see that Y, L, α and β satisfy the first LMI of (8.30). Thus, (8.30) is feasible which proves the theorem. □

Remark 8.10. If we know a priori that the noise w is contained in a given subspace, say $w(t) \in \operatorname{im} E$ for all t with E a given, known, $n \times r$ matrix, then this information can easily be exploited in the $\mathcal{H}_2$ controller design. In fact, we only need to replace the LMI involving Z by

$$\begin{bmatrix} Z & E^\top \\ E & Y \end{bmatrix} \geqslant 0.$$

Recall from Section 3.4 that prior knowledge that $w(t) \in \operatorname{im} E$ can also be captured by our noise model, see the list of special cases after Assumption 3.12. A natural choice is thus to use this given E both in the noise model, in particular using the weighting matrix (3.24), as well as in the system equation (8.16). However, we remark that this is not necessary: the noise in the experiment may come from a different subspace than the disturbances that are attenuated by the $\mathcal{H}_2$ controller.

8.2.1 Illustrative example: a fighter aircraft

In order to illustrate the theory of this section, we consider a discretised version of a continuous-time state-space model of a fighter aircraft. In particular, we consider the discrete-time (unstable) system of the form (8.15) with the true but unknown system matrices A_{true} and B_{true} given by

$$A_{\text{true}} = \begin{bmatrix} 1.000 & -0.374 & -0.190 & -0.321 & 0.056 & -0.026 \\ 0.000 & 0.982 & 0.010 & -0.000 & -0.003 & 0.001 \\ 0.000 & 0.115 & 0.975 & -0.000 & -0.269 & 0.191 \\ 0.000 & 0.001 & 0.010 & 1.000 & -0.001 & 0.001 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.741 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.741 \end{bmatrix}$$

$$B_{\text{true}} = \begin{bmatrix} 0.007 & 0.000 & -0.043 & 0.000 & 0.259 & 0.000 \\ -0.003 & 0.000 & 0.030 & 0.000 & 0.000 & 0.259 \end{bmatrix}^\top$$

respectively. We consider the performance output as in (8.19) with

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $D = 0$. First, we look for the smallest γ such that (8.21) is feasible for the given $(A_{\text{true}}, B_{\text{true}})$. This minimum value of γ is 1.000 and can be regarded as a benchmark: no data-driven method can perform better than the model-based solution using full knowledge of $(A_{\text{true}}, B_{\text{true}})$.

Of course, our goal is not to use the knowledge of $(A_{\text{true}}, B_{\text{true}})$ but to seek a data-driven solution instead. Therefore, we collect $T = 750$ input and state samples of (8.15). The entries of the inputs and initial state were drawn randomly from a Gaussian distribution with zero mean and unit variance. Also the noise samples were drawn randomly from a Gaussian distribution, with zero mean and variance σ^2 with $\sigma = 0.005$. In this example, we assume knowledge of a bound on the energy of the noise as

$$W_- W_-^\top \leqslant 1.35T\sigma^2 I. \tag{8.32}$$

We verify that this bound is satisfied for the generated noise sequence. This energy bound corresponds to chosing $\Phi_{11} = 1.35T\sigma^2 I$, $\Phi_{12} = 0$ and $\Phi_{22} = -I$ in the noise model (3.12).

Next, we want to compute an $\mathcal{H}_2$ controller for the unknown system using the generated data. We do so by minimizing γ subject to (8.30). This is a semidefinite program that we solve in Matlab, using Yalmip with Mosek as an LMI solver. The obtained controller K is given by

$$\begin{bmatrix} -0.023 & 1.413 & 0.695 & 0.227 & -1.591 & 0.090 \\ 0.001 & -0.041 & -0.028 & -0.034 & 0.010 & -2.723 \end{bmatrix}.$$

This controller stabilizes the original system $(A_{\text{true}}, B_{\text{true}})$. In addition, the system, in feedback with K, has an $\mathcal{H}_2$ performance of $\gamma_s = 1.007$. We note that this is almost identical to the smallest possible $\mathcal{H}_2$ performance of 1.000.

Subsequently, we repeat the above experiment using only a *part* of our data set. In particular, we compute an $\mathcal{H}_2$ controller via the semidefinite program as before, using only the first i samples of X_+, X_- and U_- for $i = 50, 100, \ldots, 750$. We display the results in Figure 8.1.

In each of the cases a stabilizing controller was found from data. However, the performance of these controllers when applied to the true system varies, and is quite poor for $i < 500$. Starting from $i = 500$ and onward, the performance is close to the optimal performance of the true system.

Next, we investigate what happens when we increase the variance σ^2 of the noise. First, we take $\sigma = 0.05$. We again generate 750 data samples, and assume

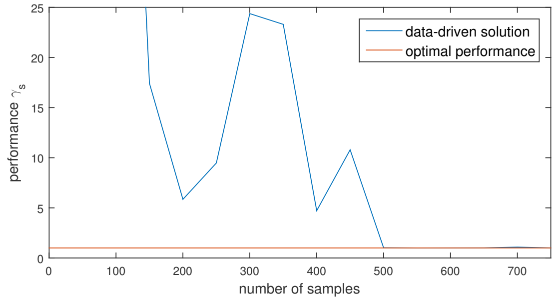


Figure 8.1: Achieved $\mathcal{H}_2$ performance of the true system in feedback with a data-based controller (top) and the optimal model-based performance of the true system (bottom).

the same bound on the noise. The $\mathcal{H}_2$ controller we obtain is given by

$$\begin{bmatrix} -0.007 & 0.179 & 0.464 & -0.284 & -1.411 & 0.100 \\ 0.005 & -0.014 & -0.363 & 0.184 & 0.123 & -1.514 \end{bmatrix}$$

and achieves a performance of $\gamma_s = 1.146$ when interconnected to the true system. Increasing the variance of the noise has the effect that the set $\Sigma_\mathcal{D}$ of systems consistent with the data becomes larger. As such, it is more difficult to control all systems in $\Sigma_\mathcal{D}$ resulting in a slightly larger γ_s. This behavior becomes even more apparent when increasing the variance of the noise to $\sigma = 0.5$. In this case we obtain the controller

$$\begin{bmatrix} -0.002 & -0.001 & 0.234 & 0.016 & -0.553 & 0.020 \\ 0.001 & -0.071 & -0.122 & -0.002 & 0.141 & -0.550 \end{bmatrix}$$

which yields a performance of $\gamma_s = 3.579$. Increasing σ even more to $\sigma = 1$ results in infeasibility of the LMIs (8.30) for any γ; the set of data-consistent systems has become too large for a quadratically stabilizing controller to exist.

We remark that the size of the set $\Sigma_\mathcal{D}$ does not only depend on the variance of the noise, but also on the available bound on the noise. Throughout this example, we have used the bound (8.32). However, if we reconsider the case of $\sigma = 0.5$ with the tighter bound $W_-W_-^\top \leqslant 1.22T\sigma^2 I$ we obtain a controller with better performance $\gamma_s = 2.706$. This illustrates the simple fact that data-driven controllers not only depend on the particular design strategy, but also on the *prior knowledge* on the noise.

We conclude the example with a remark on the dimension of the variables involved in the formulation (8.30). The symmetric matrices Y and Z both have 21 free variables. The matrix L contains 12 variables, and α and β are both scalar variables. Thus, the total number of variables is 56. The size of the largest LMI in (8.30) is 21×21.

8.3 $\mathcal{H}_\infty$ control with noisy data

In this section we study the data-driven $\mathcal{H}_\infty$ control problem. In order to do this, we will first review some basic material that will be needed in order to formulate the problem.

We will denote by $\ell_2^q(\mathbb{Z}_+)$ the linear space of all sequences v with $v(t) \in \mathbb{R}^q$ and $t \in \mathbb{Z}_+$ such that $\sum_{t=0}^{\infty} \|v(t)\|^2 < \infty$. For any such sequence v, we define its ℓ_2-norm as

$$\|v\|_2 := \left(\sum_{t=0}^{\infty} \|v(t)\|^2 \right)^{\frac{1}{2}}.$$

Next, consider the discrete-time input-state-output system

$$\begin{aligned} x(t+1) &= Ax(t) + Ew(t) \\ z(t) &= Cx(t) + Dw(t) \end{aligned} \tag{8.33}$$

with $w(t) \in \mathbb{R}^q$ and $z(t) \in \mathbb{R}^p$. Let its transfer matrix be denoted by $G(z) := C(zI - A)^{-1}E + D$. If we take as initial state $x(0) = 0$, then each input sequence w on $\mathbb{Z}_+$ yields a unique output sequence z on $\mathbb{Z}_+$. If A is stable, then this output sequence z is in $\ell_2^p(\mathbb{Z}_+)$ whenever w is in $\ell_2^q(\mathbb{Z}_+)$. The $\mathcal{H}_\infty$ performance of (8.33) is now defined as

$$J_{\mathcal{H}_\infty} := \sup_{\|w\|_2 \leqslant 1} \|z\|_2.$$

Due to the fact that A is stable, the $\mathcal{H}_\infty$ performance is indeed a finite number, and is in fact equal to the $\mathcal{H}_\infty$ norm of the transfer matrix $G(z)$, which is given by

$$\|G\|_{\mathcal{H}_\infty} := \max_{|z|=1} \|G(z)\|.$$

We now review the so-called bounded real lemma, which gives necessary and sufficient conditions for the $\mathcal{H}_\infty$ performance to be strictly less than a given tolerance.

Proposition 8.11 (Discrete-time bounded real lemma)**.** *Consider the system (8.33). Let $\gamma > 0$. Then A is stable and $J_{\mathcal{H}_\infty} < \gamma$ if and only if there exists $P > 0$ such that*

$$\begin{bmatrix} P - A^\top PA - C^\top C & -A^\top PE - C^\top D \\ -E^\top PA - D^\top C & \gamma^2 I - E^\top PE - D^\top D \end{bmatrix} > 0. \tag{8.34}$$

We will now turn to the the data-driven $\mathcal{H}_\infty$ control problem in the context of noisy data. As before, consider the true (but unknown) system

$$x(t+1) = A_{\text{true}}x(t) + B_{\text{true}}u(t) + w(t) \tag{8.35}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input and $w(t) \in \mathbb{R}^n$ is an unknown noise input. The matrices A_{true} and B_{true} denote the unknown state and input matrices. As model class $\mathcal{M}$, we take the set of all input-state systems with unknown noise inputs, with given, known, dimensions n and m, of the form

$$x(t+1) = Ax(t) + Bu(t) + w(t). \tag{8.36}$$

We assume that data (U_-, X) have been collected on the time interval $[0, T]$. Since the noise input w is assumed to be unknown, the noise samples

$$w(0), w(1), \dots, w(T-1)$$

are not measured, and are therefore not part of the data. However, we adopt the noise model of Section 3.4, and assume that the (unknown) matrix $W_- = W_{[0,T-1]}$ satisfies the quadratic matrix inequality (3.18) for a given partitioned matrix $\Phi \in \mathbf{\Pi}_{n,T}$.

As before, by Lemma 3.16 the set $\Sigma_{\mathcal{D}}$ of all systems in $\mathcal{M}$ that are consistent with the data (U_-, X) is then equal to the set of all solutions (A, B) to the inequality

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} \geqslant 0. \tag{8.37}$$

A standing assumption remains that the unknown system (8.35) is consistent with the data, i.e., is in $\Sigma_{\mathcal{D}}$, i.e., $(A_{\text{true}}, B_{\text{true}})$ satisfies the inequality (8.37).

We associate to (8.36) the output equation

$$z(t) = Cx(t) + Du(t) \tag{8.38}$$

where $z(t) \in \mathbb{R}^p$, and C and D are known matrices, chosen in order to quantify the performance. For any $(A, B) \in \Sigma_{\mathcal{D}}$, the feedback law $u = Kx$ yields the closed-loop system

$$\begin{aligned} x(t+1) &= (A+BK)x(t) + w(t) \\ z(t) &= (C+DK)x(t). \end{aligned} \tag{8.39}$$

Denote the transfer matrix of the closed loop system (8.39) by $G_K(z)$. For any K such that $A + BK$ is stable, the $\mathcal{H}_\infty$ performance associated with (8.39) is then given by

$$J_{\mathcal{H}_\infty}(K) := \|G_K\|_{\mathcal{H}_\infty}.$$

Let $\gamma > 0$. By applying Proposition 8.11 to the closed loop system (8.39), the matrix $A + BK$ is stable and $J_{\mathcal{H}_\infty}(K) < \gamma$ if and only if there exists a matrix $P > 0$ such that

$$\begin{bmatrix} P - A_K^\top P A_K - C_K^\top C_K & -A_K^\top P \\ -P A_K & \gamma^2 I - P \end{bmatrix} > 0 \tag{8.40}$$

where we have defined $A_K := A + BK$ and $C_K := C + DK$. In order to make this applicable to data-driven $\mathcal{H}_\infty$ control design, we restate (8.40) in a different form as follows.

Lemma 8.12. *Let $P > 0$. Then P satisfies the linear matrix inequality (8.40) if and only if*

$$P - A_K^\top (P^{-1} - \frac{1}{\gamma^2} I)^{-1} A_K - C_K^\top C_K > 0 \tag{8.41}$$

$$P^{-1} - \frac{1}{\gamma^2} I > 0. \tag{8.42}$$

Proof. Clearly, P satisfies (8.40) if and only if

$$P - A_K^\top \left(P + P(\gamma^2 I - P)^{-1} P \right) A_K - C_K^\top C_K > 0 \tag{8.43}$$

$$\gamma^2 I - P > 0. \tag{8.44}$$

Since $P^{-\frac{1}{2}}(\gamma^2 I - P)P^{-\frac{1}{2}} = \gamma^2(P^{-1} - \frac{1}{\gamma^2} I)$, the inequalities (8.42) and (8.44) are equivalent. Using this, it is also easily verified that $P + P(\gamma^2 I - P)^{-1} P = (P^{-1} - \frac{1}{\gamma^2} I)^{-1}$, so the left hand sides of (8.41) and (8.43) coincide. □

The above leads to the following definition of informativity for $\mathcal{H}_\infty$ control.

Definition 8.13. Let $\gamma > 0$. The data (U_-, X) are *informative for $\mathcal{H}_\infty$ control* with performance γ if there exist matrices $P > 0$ and K such that (8.41) and (8.42) hold for all $(A, B) \in \Sigma_\mathcal{D}$.

Of course, if K satisfies the conditions of Definition 8.13, then it is a suitable control gain for all $(A, B) \in \Sigma_\mathcal{D}$, in the sense that $A+BK$ is stable and $J_{\mathcal{H}_\infty}(K) < \gamma$ for all $(A, B) \in \Sigma_\mathcal{D}$.

By pre- and postmultiplication of (8.41) by P^{-1} we obtain that (8.41) and (8.42) are equivalent to

$$\begin{aligned} Y - A_{Y,L}^\top (Y - \frac{1}{\gamma^2} I)^{-1} A_{Y,L} - C_{Y,L}^\top C_{Y,L} &> 0 \\ Y - \frac{1}{\gamma^2} I &> 0 \end{aligned} \tag{8.45}$$

where we define $Y := P^{-1}$, $L := KY$, $A_{Y,L} := AY + BL$ and $C_{Y,L} := CY + DL$. Next, note that (8.45) holds if and only if

$$\begin{bmatrix} Y - C_{Y,L}^\top C_{Y,L} & A_{Y,L}^\top \\ A_{Y,L} & Y - \frac{1}{\gamma^2} I \end{bmatrix} > 0 \tag{8.46}$$

which in turn is equivalent to

$$Y - C_{Y,L}^\top C_{Y,L} > 0 \tag{8.47}$$

$$Y - \frac{1}{\gamma^2} - A_{Y,L}(Y - C_{Y,L}^\top C_{Y,L})^{-1} A_{Y,L}^\top > 0. \tag{8.48}$$

Note that (8.47) is independent of A and B. In addition, we can write (8.48) as

$$\begin{bmatrix} I \\ \hdashline A^\top \\ B^\top \end{bmatrix}^\top \left[\begin{array}{c:c} Y - \frac{1}{\gamma^2} I & 0 \\ \hdashline 0 & -\begin{bmatrix} Y \\ L \end{bmatrix} (Y - C_{Y,L}^\top C_{Y,L})^{-1} \begin{bmatrix} Y \\ L \end{bmatrix}^\top \end{array}\right] \begin{bmatrix} I \\ \hdashline A^\top \\ B^\top \end{bmatrix} > 0. \tag{8.49}$$

Observe that the inequality (8.49) is of a form where A and B appear on the left and their transposes appear on the right, analogous to (6.24) and (8.25). As before, let

$$N = \left[\begin{array}{c:c} N_{11} & N_{12} \\ \hdashline N_{12}^\top & N_{22} \end{array}\right] := \begin{bmatrix} I & X_+ \\ \hdashline 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ \hdashline 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top \tag{8.50}$$

and let M be defined by

$$M = \left[\begin{array}{c:c} M_{11} & M_{12} \\ \hdashline M_{12}^\top & M_{22} \end{array}\right] := \left[\begin{array}{c:c} Y - \frac{1}{\gamma^2} I & 0 \\ \hdashline 0 & -\begin{bmatrix} Y \\ L \end{bmatrix} (Y - C_{Y,L}^\top C_{Y,L})^{-1} \begin{bmatrix} Y \\ L \end{bmatrix}^\top \end{array}\right]. \tag{8.51}$$

Then, for given $\gamma > 0$, informativity for $\mathcal{H}_\infty$ control with performance γ holds if and only if there exist matrices $Y > 0$ and L that satisfy the inequality $Y - C_{Y,L}^\top C_{Y,L} > 0$ with in addition

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} > 0 \quad \text{for all } Z \in \mathbb{R}^{(n+m)\times n} \text{ such that } \begin{bmatrix} I \\ Z \end{bmatrix}^\top N \begin{bmatrix} I \\ Z \end{bmatrix} \geqslant 0 \tag{8.52}$$

with Z given by

$$Z := \begin{bmatrix} A^\top \\ B^\top \end{bmatrix}.$$

Moreover, in that case a suitable control gain is given by $K = LY^{-1}$.

Using the sets (A.3) and (A.12) introduced in Section A.2, condition (8.52) is equivalent to

$$\mathcal{Z}_{n+m}(N) \subseteq \mathcal{Z}^+_{n+m}(M). \tag{8.53}$$

This observation brings us in a position to apply Corollary A.23. In fact, we derive the following theorem.

Theorem 8.14. *Let $\gamma > 0$. Then the data (U_-, X) are informative for $\mathcal{H}_\infty$ control with performance γ if and only if there exist matrices $Y \in \mathbb{S}^n$, $L \in \mathbb{R}^{m\times n}$ and scalars $\alpha \geqslant 0$ and $\beta > 0$ satisfying*

$$\begin{bmatrix} Y - \frac{1}{\gamma^2}I - \beta I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y & 0 \\ 0 & 0 & 0 & L & 0 \\ 0 & Y & L^\top & Y & C_{Y,L}^\top \\ 0 & 0 & 0 & C_{Y,L} & I \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^\top \geqslant 0 \tag{8.54a}$$

$$\begin{bmatrix} Y & C_{Y,L}^\top \\ C_{Y,L} & I \end{bmatrix} > 0. \tag{8.54b}$$

Moreover, if Y and L satisfy (8.54) then $K := LY^{-1}$ is such that $A + BK$ is stable and $J_{\mathcal{H}_\infty}(K) < \gamma$ for all $(A, B) \in \Sigma_{\mathcal{D}}$.

Proof. Assume that Y and L satisfy (8.54). By the second LMI in (8.54) we have $Y > 0$. Next, compute the Schur complement of the first LMI in (8.54) with respect to the diagonal block

$$\begin{bmatrix} Y & C_{Y,L}^\top \\ C_{Y,L} & I \end{bmatrix}.$$

Of course, here we use the fact that this block is nonsingular by the second LMI of (8.54). The computation of this Schur complement yields

$$M - \alpha N \geqslant \begin{bmatrix} \beta I & 0 \\ 0 & 0 \end{bmatrix} \tag{8.55}$$

with M and N defined by (8.51) and (8.50), respectively. We thus conclude that (8.53) is satisfied. The condition $Y - C_{Y,L}^\top C_{Y,L} > 0$ holds by the second LMI of (8.54). This shows that the data (U_-, X) are informative for $\mathcal{H}_\infty$ control with performance γ and $K = LY^{-1}$ is a suitable controller.

Conversely, suppose that the data (U_-, X) are informative for $\mathcal{H}_\infty$ control with performance γ. Then there exist $Y > 0$ and L such that $Y - C_{Y,L}^\top C_{Y,L} > 0$ and the inclusion (8.53) holds. Clearly, the second LMI of (8.54) is then satisfied. To prove that the first LMI of (8.54) also holds, we want to apply Corollary A.23.

Note that we have already verified the assumptions of this theorem for the matrix N in (8.50), see the discussion preceding Theorem 3.17. In addition, we have

$$M_{22} = -\begin{bmatrix} Y \\ L \end{bmatrix} (Y - C_{Y,L}^\top C_{Y,L})^{-1} \begin{bmatrix} Y \\ L \end{bmatrix}^\top \leqslant 0$$

since $Y - C_{Y,L}^\top C_{Y,L} > 0$. Hence, Corollary A.23 is applicable. We conclude that there exist $\alpha \geqslant 0$ and $\beta > 0$ such that (8.55) holds. Using the same Schur complement argument as above, we see that Y, L, α and β satisfy the first LMI of (8.54). Thus, (8.54) is feasible. This completes the proof of the theorem. $\square$

If Y and L satisfy (8.54) then $K := LY^{-1}$ and $P := Y^{-1}$ satisfy (8.40) for all (A, B) in the set $\Sigma_{\mathcal{D}}$ of systems consistent with the data. Clearly, (8.40) implies that

$$x(t+1)^\top P x(t+1) - x(t)^\top P x(t) \leqslant \begin{bmatrix} w(t) \\ z(t) \end{bmatrix}^\top \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} w(t) \\ z(t) \end{bmatrix}$$

for all $t \in \mathbb{R}$, where x, w and z satisfy the closed loop system equations (8.39). This can be interpreted as saying that the system (8.39) is dissipative with respect to the supply rate

$$s(w, z) := \begin{bmatrix} w \\ z \end{bmatrix}^\top \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}.$$

with storage function $x^\top P x$. In other words: the control law $u = Kx$ with $K := LY^{-1}$ makes all systems in $\Sigma_{\mathcal{D}}$ dissipative with *common* storage function given by $P := Y^{-1}$.

8.4 Notes and references

The solution to the data-driven suboptimal $\mathcal{H}_2$ problem in Section 8.1 is based on the paper [176]. This result relies on an LMI characterization under which a given feedback gain is suboptimal, see Proposition 7.10. In order to prove this proposition, we have used the auxiliary Lemma 7.11 on the uniqueness of solutions to (discrete-time) Lyapunov equations. A proof of this lemma can be found in [12, Thm. 4.50].

The treatment of the data-driven $\mathcal{H}_2$ control problem in the setting of noisy data (Section 8.2) is based on the paper [169]. In the case of noise, we have used the matrix version of the S-lemma in Corollary A.23 to derive necessary and sufficient conditions for informativity for $\mathcal{H}_2$ control in terms of linear matrix inequalities. Similar to the setting of noise-free data, the number of involved decision variables is independent of the time horizon of the experiment. We have exploited this fact by applying our results to a dataset consisting of 750 samples obtained from a discrete-time model of a fighter aircraft. In fact, in Subsection 8.2.1 we have considered the discretization of the model used in [155, Ex. 10.1.2]. To obtain this discrete-time model, the system from [155] was discretized using a sampling time of 0.01.

We refer to the paper [47] for the solution to the $\mathcal{H}_\infty$ control problem for state-space systems, requiring the existence of two solutions of Riccati equations that satisfy a coupling condition. In Section 8.3 we have solved a data-driven version of the $\mathcal{H}_\infty$ control problem. An important tool to achieve this, is the discrete-time *bounded real lemma* that provides an LMI condition under which the $\mathcal{H}_\infty$ norm of a transfer matrix is less than a given tolerance. We refer to the book [155] for more details on this result. For the data-driven solution to the $\mathcal{H}_\infty$ control problem, we refer to [169] and [173].

9

Data-driven analysis and design using input-output data

In this chapter we will study data-driven analysis and control design for discrete-time input-output systems described by higher order difference equations. In particular this means that we will temporarily leave the realm of input-state and input-state-output systems and instead consider linear systems in autoregressive (AR) form. Before turning to the subject of data-driven analysis and control for this class of systems, we will first give a brief review of systems in autoregressive form in Section 9.1. An important role in this chapter will be played by quadratic difference forms. A brief introduction to this topic will be provided in Section 9.2.

After the discussions in these two introductory sections, we will turn to the main subject of this chapter. We will discuss unknown AR systems and data in Section 9.3, and the special case of uncontrolled AR systems in Section 9.3.1. In Section 9.4 we will treat the stability analysis of autonomous AR systems. Based on the results in that section, in Section 9.5 we will study the problem of data-driven stability analysis of autonomous systems. Subsequently, data-driven stabilization of input-output systems will be considered in Sections 9.6 and 9.7. We will close this chapter with a numerical example in Section 9.8.

9.1 Systems represented by AR models

In this section we review some basic material on systems represented by autoregressive (AR) models of the form

$$\begin{aligned} &y(t+L) + P_{L-1}y(t+L-1) + \cdots + P_1 y(t+1) + P_0 y(t) = \\ &Q_L u(t+L) + Q_{L-1}u(t+L-1) + \cdots + Q_1 u(t+1) + Q_0 u(t). \end{aligned} \tag{9.1}$$

Here L is a positive integer, called the *order* of the system. The system variables $u(t)$ and $y(t)$ are assumed to take their values in $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively. The parameters of the model are real $p \times p$ matrices $P_0, P_1, \ldots, P_{L-1}$ and $p \times m$ matrices $Q_0, Q_1, \ldots, Q_L$. By defining the *shift operator* σ as

$$(\sigma f)(t) = f(t+1) \tag{9.2}$$

the AR equation (9.1) can be written as

$$P(\sigma)y = Q(\sigma)u \tag{9.3}$$

where $P(\xi)$ and $Q(\xi)$ are the real $p \times p$ and $p \times m$ polynomial matrices defined by

$$\begin{aligned} P(\xi) &= I\xi^L + P_{L-1}\xi^{L-1} + \cdots + P_1\xi + P_0 \\ Q(\xi) &= Q_L\xi^L + Q_{L-1}\xi^{L-1} + \cdots + Q_1\xi + Q_0. \end{aligned} \tag{9.4}$$

The polynomial matrix $P(\xi)$ is nonsingular, equivalently $\det P(\xi)$ is not the zero polynomial, and the rational matrix $P^{-1}(\xi)Q(\xi)$ is proper. Hence (9.1) represents a causal input-output system with input u and output y. We will use the notation

$$R(\xi) := \begin{bmatrix} -Q(\xi) & P(\xi) \end{bmatrix} \text{ and } w := \begin{bmatrix} u \\ y \end{bmatrix}. \tag{9.5}$$

The equation (9.3) can then be rewritten as

$$R(\sigma)w = 0. \tag{9.6}$$

Clearly, $R(\xi)$ is a real $p \times q$ polynomial matrix with $q := p + m$. Often, (9.6) is called a *kernel representation* of the input-output system (9.1). The linear space of all solutions $w : \mathbb{Z}_+ \to \mathbb{R}^q$ of (9.6) is called the *behavior* of the system, and will be denoted by $\mathcal{B}(R)$. In the special case that $m = 0$, i.e. there are no inputs, the polynomial matrix $Q(\xi)$ is void and $R(\xi) = P(\xi)$. In that case (9.6) represents an *autonomous* system and the associated behavior, denoted by $\mathcal{B}(P)$, is a finite-dimensional linear space.

Note that

$$R(\xi) = \begin{bmatrix} -Q_L & I \end{bmatrix} \xi^L + R_{L-1}\xi^{L-1} + \cdots + R_1\xi + R_0$$

where $R_i := \begin{bmatrix} -Q_i & P_i \end{bmatrix}$ for $i = 0, 1, \ldots, L-1$. We collect the matrices R_i and $-Q_L$ in the real $p \times (qL + m)$ matrix

$$R := \begin{bmatrix} R_0 & R_1 & \cdots & R_{L-1} & -Q_L \end{bmatrix}. \tag{9.7}$$

This matrix will be called the coefficient matrix of $R(\xi)$. Note that it does not include the identity matrix appearing in $\begin{bmatrix} -Q_L & I \end{bmatrix}$.

For any given $T > 0$, we define the behavior of $\mathcal{B}(R)$ restricted to the interval $[0, T]$ by

$$\mathcal{B}(R)|_{[0,T]} := \{ w_{[0,T]} \in \mathbb{R}^{q(T+1)} \mid w \in \mathcal{B}(R) \}.[1]$$

Then the following holds.

[1]For the notation $w_{[i,j]}$ and (later on) $u_{[i,j]}$, $y_{[i,j]}$ we refer to Subsection 1.2.1.

Lemma 9.1. *Consider the system* (9.6) *with order* L. *Let* $\mathcal{B}(R)$ *be its behavior. Then we have*

$$\mathcal{B}(R)|_{[0,L-1]} = \mathbb{R}^{qL}$$

and

$$\mathcal{B}(R)|_{[0,L]} = \ker \begin{bmatrix} R & I \end{bmatrix} = \operatorname{im} \begin{bmatrix} I \\ -R \end{bmatrix}.$$

Proof. To prove the first statement, let $w_0, w_1, \ldots, w_{L-1} \in \mathbb{R}^q$. We need to prove that there exists $w \in \mathcal{B}(R)$ such that $w(t) = w_t$ for $t \in [0, L-1]$. It suffices to prove that there exist $w_L \in \mathbb{R}^q$ satisfying

$$\begin{bmatrix} R_0 & R_1 & \cdots & R_{L-1} & \begin{bmatrix} -Q_L & I \end{bmatrix} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_L \end{bmatrix} = 0.$$

Since $\begin{bmatrix} -Q_L & I \end{bmatrix}$ has full row rank, such w_L clearly exists.

To prove the second statement, note that the inclusion $\mathcal{B}(R)|_{[0,L]} \subseteq \ker \begin{bmatrix} R & I \end{bmatrix}$ is immediate. To prove the reverse inclusion, let $w_0, w_1, \ldots, w_L \in \mathbb{R}^q$ be such that

$$\begin{bmatrix} R_0 & R_1 & \cdots & R_{L-1} & \begin{bmatrix} -Q_L & I \end{bmatrix} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_L \end{bmatrix} = 0.$$

We will prove that there exists $w \in \mathcal{B}(R)$ such that $w(t) = w_t$ for $t = 0, 1, \ldots, L$. For this, it suffices to prove that there exist $w_{L+1} \in \mathbb{R}^q$ satisfying

$$\begin{bmatrix} R_0 & R_1 & \cdots & R_{L-1} & \begin{bmatrix} -Q_L & I \end{bmatrix} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{L+1} \end{bmatrix} = 0.$$

Again, since $\begin{bmatrix} -Q_L & I \end{bmatrix}$ has full row rank such w_{L+1} exists. The claim that the kernel and the image are equal is immediate. This completes the proof of the lemma. □

Note that the above lemma also applies to the autonomous case, in which $Q(\xi)$ is void. In that case the coefficient matrix R of $R(\xi)$ as in (9.7) is equal to the matrix $\begin{bmatrix} P_0 & P_1 & \cdots & P_{L-1} \end{bmatrix}$.

9.2 Quadratic difference forms

In the stability analysis of systems represented by AR models, an important role is played by quadratic difference forms. In the present section we will introduce these, and discuss some important properties.

Let N and q be positive integers and for $i, j \in [0, N]$ let $\Phi_{i,j} \in \mathbb{R}^{q \times q}$ be such that $\Phi_{i,i} \in \mathbb{S}^q$ and $\Phi_{i,j} = \Phi_{j,i}^\top$ for all $i \neq j$. Arrange these matrices into the partitioned matrix $\Phi \in \mathbb{S}^{(N+1)q}$ given by

$$\Phi := \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots & \Phi_{0,N} \\ \Phi_{1,0} & \Phi_{1,1} & \cdots & \Phi_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{N,0} & \Phi_{N,1} & \cdots & \Phi_{N,N} \end{bmatrix}$$

Then the *quadratic difference form* (QDF) associated with Φ is the operator Q_Φ that maps $\mathbb{R}^q$-valued functions w on $\mathbb{Z}_+$ to $\mathbb{R}$-valued functions $Q_\Phi(w)$ on $\mathbb{Z}_+$ defined by

$$Q_\Phi(w)(t) := \sum_{k,\ell=0}^{N} w(t+k)^\top \Phi_{k,\ell}\, w(t+\ell). \tag{9.8}$$

In terms of the matrix Φ this can be written as

$$Q_\Phi(w)(t) = w_{[t,t+N]}^\top\, \Phi\, w_{[t,t+N]}.$$

We define the *degree* of the QDF (9.8) as the smallest integer d such that $\Phi_{ij} = 0$ for all pairs (i, j) with $i > d$. The matrix Φ is called a coefficient matrix of the QDF. Note that a given QDF does not determine the coefficient matrix uniquely. However, if the degree of the QDF is d, it allows a coeficient matrix $\Phi \in \mathbb{S}^{(d+1)q}$.

The QDF Q_Φ is called nonnegative if $Q_\Phi(w) \geqslant 0$ for all $w : \mathbb{Z} \to \mathbb{R}^q$. We denote this as $Q_\Phi \geqslant 0$. Clearly, this holds if and only if $\Phi \geqslant 0$. The QDF is called positive if it is nonnegative and, in addition, $Q_\Phi(w) = 0$ implies $w = 0$. This is denoted as $Q_\Phi > 0$. Likewise we define nonpositivity and negativity.

For a given QDF Q_Φ, its *rate of change* along a given $w : \mathbb{Z} \to \mathbb{R}^q$ is given by $Q_\Phi(w)(t+1) - Q_\Phi(w)(t)$. It turns out that the rate of change defines a QDF itself. Indeed, by defining the matrix $\nabla\Phi \in \mathbb{S}^{(N+2)q}$ by

$$\nabla\Phi := \begin{bmatrix} 0_q & 0 \\ 0 & \Phi \end{bmatrix} - \begin{bmatrix} \Phi & 0 \\ 0 & 0_q \end{bmatrix} \tag{9.9}$$

it is easily verified that

$$Q_{\nabla\Phi}(w)(t) = Q_\Phi(w)(t+1) - Q_\Phi(w)(t)$$

for all $w : \mathbb{Z}_+ \to \mathbb{R}^q$ and $t \in \mathbb{Z}_+$.

Quadratic difference forms are particularly relevant in combination with AR systems. Let $R(\xi)$ be a real $p \times q$ polynomial matrix and consider the AR system

$$R(\sigma)w = 0.$$

Let $\mathcal{B}(R)$ be the behavior of this system. The QDF Q_Φ is called nonnegative on $\mathcal{B}(R)$ if $Q_\Phi(w) \geqslant 0$ for all $w \in \mathcal{B}(R)$. It is called positive on $\mathcal{B}(R)$ if it is nonnegative on $\mathcal{B}(R)$ and, in addition, $Q_\Phi(w) = 0$ for $w \in \mathcal{B}(R)$ implies $w = 0$. We denote this as $Q_\Phi \geqslant 0$ on $\mathcal{B}(R)$ and $Q_\Phi > 0$ on $\mathcal{B}(R)$, respectively. Likewise we define nonpositivity and negativity on $\mathcal{B}(R)$.

Two given QDFs Q_{Φ_1} and Q_{Φ_2} are called $\mathcal{B}(R)$-*equivalent* if they coincide on solutions of $R(\sigma)w = 0$, i.e. $Q_{\Phi_1}(w) = Q_{\Phi_2}(w)$ for all $w \in \mathcal{B}(R)$. This is denoted as $Q_{\Phi_1} = Q_{\Phi_2}$ on $\mathcal{B}(R)$.

It turns out that for any QDF that is nonnegative on a given input-output behavior, there exists an equivalent QDF that is nonnegative, equivalently, its coefficient matrix is positive semidefinite.

Theorem 9.2. *Consider the input-output system (9.3), with $m > 1$ and $P(\xi)$ and $Q(\xi)$ polynomial matrices of the form (9.4). Let $R(\xi)$ and w be as in (9.5). For any QDF $Q_{\Phi'}$ such that $Q_{\Phi'} \geqslant 0$ on $\mathcal{B}(R)$ there exists a QDF Q_Φ such that $Q_\Phi = Q_{\Phi'}$ on $\mathcal{B}(R)$ and $Q_\Phi \geqslant 0$, equivalently $\Phi \geqslant 0$.*

Proof. Let $d = \deg(Q_{\Phi'})$. Let the columns of the real matrix K form a basis for $\mathcal{B}(R)\,|_{[0,d]}$. Since $Q_\Phi \geqslant 0$ on $\mathcal{B}(R)$ we have $K^\top \Phi' K \geqslant 0$. Therefore, there exists a real matrix C such that $K^\top \Phi' K = C^\top C$. Clearly, $\ker K \subseteq \ker C$ and therefore $\operatorname{im} C^\top \subseteq \operatorname{im} K^\top$. We conclude that there exists a real matrix F such that $C^\top = K^\top F^\top$. As such, $K^\top \Phi' K = K^\top F^\top F K$. Finally, we see that $\Phi := F^\top F \geqslant 0$ and $Q_\Phi = Q_{\Phi'}$ on $\mathcal{B}(R)$. □

In the autonomous case, i.e. $m = 0$, a stronger result holds. In that case, every QDF turns out to be equivalent to a QDF with degree at most the order of the system. Indeed, let $P(\xi)$ be a square polynomial matrix as in (9.4), with corresponding autonomous system $P(\sigma)y = 0$ of order L. Denote its behavior by $\mathcal{B}(P)$. Then we have

Lemma 9.3. *For any QDF $Q_{\Phi'}$ there exists a QDF Q_Φ with degree at most $L - 1$ such that $Q_\Phi(y) = Q_{\Phi'}(y)$ for all $y \in \mathcal{B}(P)$. In addition, if $Q_{\Phi'} \geqslant 0$ on $\mathcal{B}(P)$ then $Q_\Phi \geqslant 0$, equivalently, $\Phi \geqslant 0$.*

Proof. Let $d = \deg(Q_{\Phi'})$ and let $y \in \mathcal{B}(P)$. Then, we have

$$Q_{\Phi'}(y)(t) = \sum_{k=0}^{d} \sum_{l=0}^{d} y^\top(t+k)\Phi'_{k,l}\, y(t+l). \tag{9.10}$$

First consider the case that $d \leqslant L-1$. Since $\mathcal{B}(P)\mid_{[0,L-1]}=\mathbb{R}^{pL}$, we readily have that if $Q_{\Phi'} \geqslant 0$ on $\mathcal{B}(P)$ then $\Phi' \geqslant 0$. Next, suppose that $d \geqslant L$. Let k be such that $L \leqslant k \leqslant d$. Then we have

$$y(t+k) = -P_{L-1}y(t+k-1) - \cdots - P_0 y(t+k-L).$$

Therefore, one can substitute $y(t+k)$ for $k \in [L,d]$ into (9.10) to obtain

$$Q_{\Phi'}(y)(t) = \sum_{k=0}^{L-1}\sum_{l=0}^{L-1} y^\top(t+k)\Phi_{k,l}y(t+l). \tag{9.11}$$

where $\Phi_{i,j}$ are suitable $p \times p$ matrices. Define

$$\Phi := \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots & \Phi_{0,L-1} \\ \Phi_{1,0} & \Phi_{1,1} & \cdots & \Phi_{1,L-1} \\ \vdots & \vdots & & \vdots \\ \Phi_{L-1,0} & \Phi_{L-1,1} & \cdots & \Phi_{L-1,L-1} \end{bmatrix}.$$

It follows from (9.11) that $Q_\Phi(y) = Q_{\Phi'}(y)$ for all $y \in \mathcal{B}(P)$. Moreover, again by the fact that $\mathcal{B}(P)\mid_{[0,L-1]}=\mathbb{R}^{pL}$, we see that if $Q_\Phi \geqslant 0$ on $\mathcal{B}(P)$ we have $\Phi \geqslant 0$, equivalently, $Q_\Phi \geqslant 0$.

□

9.3 Input-output AR systems and data

In this chapter we will in particular consider input-output systems with noise, represented by autoregressive (AR) models of the form

$$\begin{aligned} &y(t+L) + P_{L-1}y(t+L-1) + \cdots + P_1 y(t+1) + P_0 y(t) = \\ &Q_L u(t+L) + Q_{L-1}u(t+L-1) + \cdots + Q_1 u(t+1) + Q_0 u(t) + v(t). \end{aligned} \tag{9.12}$$

Here L is a positive integer, again called the order. The control input $u(t)$ and output $y(t)$ are assumed to take their values in $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively. The term $v(t)$ represents unknown noise. The parameters of the model are real $p \times p$ matrices $P_0, P_1, \ldots, P_{L-1}$ and $p \times m$ matrices $Q_0, Q_1, \ldots, Q_L$. As we already saw in Section 9.1, using the shift operator $(\sigma f)(t) = f(t+1)$ the difference equation (9.12) can be written as

$$P(\sigma)y = Q(\sigma)u + v \tag{9.13}$$

where $P(\xi)$ and $Q(\xi)$ are the real $p \times p$ and $p \times m$ polynomial matrices defined by

$$\begin{aligned} P(\xi) &= I\xi^L + P_{L-1}\xi^{L-1} + \cdots + P_1\xi + P_0 \\ Q(\xi) &= Q_L\xi^L + Q_{L-1}\xi^{L-1} + \cdots + Q_1\xi + Q_0. \end{aligned} \tag{9.14}$$

Note that the leading coefficient matrix of $P(\xi)$ is the $p \times p$ identity matrix. This immediately implies that $P(\xi)$ is nonsingular and that $P^{-1}(\xi)Q(\xi)$ is proper. Thus, indeed, (9.13) represents a causal input-output system with control input u, noise input v and output y.

Next, we will return to our context of data-driven analysis and control. We will deal with analysis and control design for systems of the form (9.13), where the polynomial matrices $P(\xi)$ and $Q(\xi)$ are unknown. The order L and the dimensions m and p are assumed to be known. We assume that we have obtained noisy input-output data on a given finite time interval. These data are generated by an underlying true (but unknown) system. In the special case that this unknown system has no control inputs then we only have output data, and we want to use these to check whether the system is stable. On the other hand, in case that control inputs are present we want to to use the input-output data to check whether there exists a stabilizing feedback controller and, if so, determine such controller using only the data. In the present section we will focus on the case that control inputs are present, i.e. the situation that $m > 0$.

As stated above, we assume that we have noisy input-output data

$$(u_{[0,T]}, y_{[0,T]}) \tag{9.15}$$

on a given time interval $[0, T]$, with $T \geqslant L$. These noisy data are obtained from the true system. Assume that this true system is represented by (unknown) polynomial matrices $P_{\text{true}}(\xi)$ and $Q_{\text{true}}(\xi)$ of the form (9.14). In other words, the true system is represented by the equation $P_{\text{true}}(\sigma)y = Q_{\text{true}}(\sigma)u + v$.

More concretely, we assume that $(u_{[0,T]}, y_{[0,T]})$ are samples on the interval $[0, T]$ of u and y that satisfy

$$P_{\text{true}}(\sigma)y = Q_{\text{true}}(\sigma)u + v$$

for some unknown noise signal v. We do make the following assumption on the noise v during the sampling interval.

Assumption 9.4. The noise samples $v_{[0,T-L]}$, collected in the real $p \times (T-L+1)$ matrix $V := V_{[0,T-L]}$ satisfy the quadratic matrix inequality

$$\begin{bmatrix} I \\ V^\top \end{bmatrix}^\top \Pi \begin{bmatrix} I \\ V^\top \end{bmatrix} \geqslant 0 \tag{9.16}$$

where $\Pi \in \mathbb{S}^{p+T-L+1}$ is a known partitioned matrix

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}$$

with $\Pi_{11} \in \mathbb{S}^p$, $\Pi_{12} \in \mathbb{R}^{p \times (T-L+1)}$, $\Pi_{21} = \Pi_{12}^\top$ and $\Pi_{22} \in \mathbb{S}^{T-L+1}$. We assume that $\Pi \in \mathbf{\Pi}_{p,T-L+1}$ with, in addition, $\Pi_{22} < 0$. In particular this implies that the set $\mathcal{Z}_{T-L+1}(\Pi)$ of matrices V that satisfy (9.16) is nonempty, convex and bounded (see Theorem A.5).

Assumption (9.4) on the noise samples $v(0), \dots, v(T-L)$ can capture various types of bounds. For examples we refer the reader to Section 3.4.

Now define $q := p + m$ and denote the unknown $p \times q$ polynomial matrix $\begin{bmatrix} -Q(\xi) & P(\xi) \end{bmatrix}$ by $R(\xi)$. Also denote

$$w := \begin{bmatrix} u \\ y \end{bmatrix}.$$

Then (9.13) can be written as

$$R(\sigma)w = v. \tag{9.17}$$

Collect the (unknown) coefficient matrices of $R(\xi)$ in the $p \times (qL + m)$ matrix

$$R := \begin{bmatrix} -Q_0 & P_0 & -Q_1 & P_1 & \cdots & -Q_{L-1} & P_{L-1} & -Q_L \end{bmatrix} \tag{9.18}$$

Note that, with a slight abuse of notation, we denote both the polynomial matrix and its coefficient matrix by R. We also arrange the data $(u_{[0,T]}, y_{[0,T]})$ into the vectors

$$w(t) = \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \quad t \in [0, T]$$

and define the associated depth $L + 1$ Hankel matrix by

$$H(w) := \begin{bmatrix} w(0) & w(1) & \cdots & w(T-L) \\ w(1) & w(2) & \cdots & w(T-L+1) \\ \vdots & \vdots & & \vdots \\ w(L) & w(L+1) & \cdots & w(T) \end{bmatrix}. \tag{9.19}$$

Furthermore, we partition

$$H(w) = \begin{bmatrix} H_1(w) \\ H_2(w) \end{bmatrix} \tag{9.20}$$

where $H_1(w)$ contains the first $qL + m$ rows and $H_2(w)$ the last p rows. It is then easily verified that any input-output system (9.17) for which the coefficient matrix R defined in (9.18) satisfies

$$\begin{bmatrix} R & I \end{bmatrix} \begin{bmatrix} H_1(w) \\ H_2(w) \end{bmatrix} = V \tag{9.21}$$

for some $V \in \mathcal{Z}_{T-L+1}(\Pi)$, could have generated the noisy input-output data (9.15). More precisely, $w(0), w(1), \dots, w(T)$ are also samples on the interval $[0, T]$ of a w that satisfies

$$R(\sigma)w = v$$

for some v satisfying Assumption 9.4. Therefore, if R satisfies (9.21) for some $V \in \mathcal{Z}_{T-L+1}(\Pi)$, we call the AR system corresponding to the matrix R *consistent with the data.* Recall that, in particular, the true system is consistent with the data. Now define

$$N := \begin{bmatrix} I & H_2(w) \\ 0 & H_1(w) \end{bmatrix} \Pi \begin{bmatrix} I & H_2(w) \\ 0 & H_1(w) \end{bmatrix}^\top . \tag{9.22}$$

Then by combining (9.16) and (9.21) we see that the system corresponding to the matrix R is consistent with the data if and only if $R^\top$ satisfies the QMI

$$\begin{bmatrix} I \\ R^\top \end{bmatrix}^\top N \begin{bmatrix} I \\ R^\top \end{bmatrix} \geqslant 0 \tag{9.23}$$

equivalently

$$R^\top \in \mathcal{Z}_{qL+m}(N).$$

Since the true system is consistent with the data, the set $\mathcal{Z}_{qL+m}(N)$ is nonempty.

9.3.1 Uncontrolled AR systems and data

In this section we consider the special case that the unknown system (9.13) has no control inputs, i.e. $m = 0$. In that case we only have output data and (9.12) reduces to

$$y(t+L) + P_{L-1}y(t+L-1) + \cdots + P_1 y(t+1) + P_0 y(t) = v(t) \tag{9.24}$$

and (9.13) to

$$P(\sigma)y = v \tag{9.25}$$

with $P(\xi)$ a nonsingular polynomial matrix. We will now briefly discuss the notion of noisy data for this special case. In fact, in this case we have only output data $y_{[0,T]}$ on a finite time-interval $[0, T]$ with $T \geqslant L$. We assume that these data come from an unknown true system. Suppose this true system is represented by the unknown polynomial matrix $P_{\text{true}}(\xi)$, with $P_{\text{true}}(\xi)$ of the form (9.14). The true system dynamics are then given by $P_{\text{true}}(\sigma)y = v$. Again we assume that the noise v is unknown, but on the time interval $[0, T-L]$ its samples satisfy Assumption 9.4.

Any system in the model class of systems of the form (9.24) with fixed dimension p and order L is parametrized by its coefficient matrices $P_0, P_1, \ldots, P_{L-1}$. We collect these matrices in the $p \times pL$ matrix

$$P := \begin{bmatrix} P_0 & P_1 & \cdots & P_{L-1} \end{bmatrix}. \tag{9.26}$$

Recalling that there are no control inputs, we have $w = y$. Therefore we denote the Hankel matrix associated with the data as given by (9.19) by $H(y)$ and as before partition this matrix as

$$H(y) = \begin{bmatrix} H_1(y) \\ H_2(y) \end{bmatrix}$$

where $H_1(y)$ contains the first pL rows and $H_2(y)$ the last p rows. Also define

$$N := \begin{bmatrix} I & H_2(y) \\ 0 & H_1(y) \end{bmatrix} \Pi \begin{bmatrix} I & H_2(y) \\ 0 & H_1(y) \end{bmatrix}^\top . \tag{9.27}$$

Then, as in Section 9.3, the system (9.25) with coefficient matrices collected in the matrix P is consistent with the data if and only if

$$\begin{bmatrix} I \\ P^\top \end{bmatrix}^\top N \begin{bmatrix} I \\ P^\top \end{bmatrix} \geqslant 0 \tag{9.28}$$

equivalently, $P^\top \in \mathcal{Z}_{pL}(N)$. Since the true system is assumed to be consistent with the data, the set $\mathcal{Z}_{pL}(N)$ is nonempty.

9.4 Stability of autonomous AR systems

In this section we review some facts on stability and Lyapunov theory in the context of autonomous systems represented by AR models. We first define stability.

Definition 9.5. Let $P(\xi)$ be a nonsingular polynomial matrix. The corresponding autonomous system $P(\sigma)y = 0$ is called *stable* if $y(t) \to 0$ as $t \to \infty$ for all solutions y on $\mathbb{Z}_+$.

Recall from Section 9.1 that the behavior, i.e. the space of all solutions of $P(\sigma)y = 0$ on $\mathbb{Z}_+$, is denoted by $\mathcal{B}(P)$. Stability of autonomous AR systems can be characterized in terms of quadratic difference forms. In fact, the following proposition holds. For its proof, we refer to [90, Thm. 1].

Proposition 9.6. *Let $P(\xi)$ be a nonsingular polynomial matrix. Furthermore, consider any QDF Q_Φ such that $Q_\Phi < 0$ on $\mathcal{B}(P)$. The autonomous system $P(\sigma)y = 0$ is stable if and only if there exists a QDF Q_Ψ such that $Q_\Psi \geqslant 0$ on $\mathcal{B}(P)$ and $Q_{\nabla\Psi} = Q_\Phi$ on $\mathcal{B}(P)$.*

For obvious reasons, we refer to the QDF Q_Ψ as a *Lyapunov function*. In principle, the above theorem does not specify the degree of Q_Ψ which could be large. However, it turns out that if $P(\xi)$ is of the form

$$P(\xi) = I\xi^L + P_{L-1}\xi^{L-1} + \cdots + P_1\xi + P_0 \tag{9.29}$$

and the corresponding system $P(\sigma)y = 0$ of order L is stable, then there exists a Lyapunov function of degree at most $L - 1$.

Lemma 9.7. *Let $P(\xi)$ be a polynomial matrix of the form (9.29). Let Q_Φ be any QDF of degree L such that $Q_\Phi < 0$ on $\mathcal{B}(P)$. The autonomous system $P(\sigma)y = 0$ is stable if and only if there exists a QDF Q_Ψ of degree at most $L-1$ such that $Q_\Psi \geqslant 0$, and $Q_{\nabla\Psi} = Q_\Phi$ on $\mathcal{B}(P)$.*

Proof. We only need to prove the 'only if' direction. By Proposition 9.6, there exists a QDF $Q_{\Psi'}$ such that $Q_{\Psi'} \geqslant 0$ on $\mathcal{B}(P)$ and $Q_{\nabla\Psi'} = Q_\Phi$ on $\mathcal{B}(P)$. By Lemma 9.3 there exists a QDF Q_Ψ of degree at most $L - 1$ that is $\mathcal{B}(P)$-equivalent to $Q_{\Psi'}$ and that satisfies $Q_\Psi \geqslant 0$. Finally, $Q_{\nabla\Psi}$ and $Q_{\nabla\Psi'}$ are also $\mathcal{B}(P)$-equivalent and therefore $Q_{\nabla\Psi} = Q_\Phi$ on $\mathcal{B}(P)$. □

9.5 Data-driven stability of autonomous AR systems

In this section we study data-based stability *analysis* for systems of the form (9.25). By stability of this system we mean that if the noise vanishes, i.e. $v = 0$, then all solutions y tend to zero as time tends to infinity, equivalently, the corresponding autonomous system $P(\sigma)y = 0$ is stable. Our aim is to develop a test that determines whether our true system is stable on the basis of the output data $y_{[0,T]}$. As we saw in Subsection 9.3.1, the data do not necessarily determine the true system uniquely. Thus we are forced to test stability for all systems that are consistent with the data, that is for all systems for which the corresponding matrix P (see (9.26)) is in $\mathcal{Z}_{pL}(N)$, where N given by (9.27).

In order to proceed, we will first express the existence of a Lyapunov function Q_Ψ for the autonomous system $P(\sigma)y = 0$ in terms of a quadratic matrix inequality. This QMI involves a symmetric matrix Ψ of dimensions $pL \times pL$ leading to a Lyapunov function Q_Ψ and the matrix $P = \begin{bmatrix} P_0 & P_1 & \cdots & P_{L-1} \end{bmatrix}$. Indeed, we have:

Theorem 9.8. *Let $P(\xi) = I\xi^L + P_{L-1}\xi^{L-1} + \cdots + P_1\xi + P_0$ and let $P(\sigma)y = 0$ be the corresponding autonomous system. This system is stable if and only if there exists $\Psi \in \mathbb{S}^{pL}$ such that $\Psi \geqslant 0$ and*

$$\begin{bmatrix} I \\ -P \end{bmatrix}^\top \left(\begin{bmatrix} 0_p & 0 \\ 0 & \Psi \end{bmatrix} - \begin{bmatrix} \Psi & 0 \\ 0 & 0_p \end{bmatrix} \right) \begin{bmatrix} I \\ -P \end{bmatrix} < 0. \tag{9.30}$$

Any such Ψ defines a Lyapunov function Q_Ψ.

Proof. We first prove the 'if' part by showing that the QDF Q_Ψ associated with the matrix Ψ is a Lyapunov function. Since $\Psi \geqslant 0$ we have $Q_\Psi \geqslant 0$ so by Proposition 9.6 it suffices to show that $Q_{\nabla\Psi} < 0$ on $\mathcal{B}(P)$. Following (9.9), denote the matrix in the middle of (9.30) by $\nabla\Psi$. Let $y \in \mathcal{B}(P)$. Then, for all $t \in \mathbb{Z}_+$ we have

$$y(t+L) + P_{L-1}y(t+L-1) + \ldots + P_1 y(t+1) + P_0 y(t) = 0.$$

This implies that

$$y_{[t,t+L]} = \begin{bmatrix} I \\ -P \end{bmatrix} y_{[t,t+L-1]}$$

for all $t \in \mathbb{Z}_+$. Thus we compute

$$\begin{aligned} Q_{\nabla\Psi}(y)(t) &= y_{[t,t+L]}^\top \, \nabla\Psi \, y_{[t,t+L]} \\ &= y_{[t,t+L-1]}^\top \begin{bmatrix} I \\ -P \end{bmatrix}^\top \nabla\Psi \begin{bmatrix} I \\ -P \end{bmatrix} y_{[t,t+L-1]} \end{aligned}$$

which implies $Q_{\nabla\Psi}(y)(t) \leqslant 0$ for all $t \in \mathbb{Z}_+$ and $Q_{\nabla\Psi}(y)(t) = 0$ for all $t \in \mathbb{Z}_+$ if and only if $y(t) = 0$ for all $t \in \mathbb{Z}_+$. This shows that $Q_{\nabla\Psi} < 0$ on $\mathcal{B}(P)$.

Next, we turn to proving the 'only if' part. Suppose the system is stable. Define $\Phi = -I_{p(L+1)}$ so that obviously $Q_\Phi < 0$ on $\mathcal{B}(P)$. According to Lemma 9.7 there exists $\Psi \in \mathbb{S}^{pL}$ with $\Psi \geqslant 0$ such that $Q_{\nabla\Psi} = Q_\Phi$ on $\mathcal{B}(P)$. We claim that Ψ satisfies (9.30). Indeed, take any $y_0, y_1, \ldots, y_{L-1}$ not all equal to zero. Clearly, there exists $y \in \mathcal{B}(P)$ such that $y(t) = y_t$, $t \in [0, L-1]$. Then,

$$\begin{bmatrix} y_0 \\ \vdots \\ y_{L-1} \end{bmatrix}^\top \begin{bmatrix} I \\ -P \end{bmatrix}^\top \nabla\Psi \begin{bmatrix} I \\ -P \end{bmatrix} \begin{bmatrix} y_0 \\ \vdots \\ y_{L-1} \end{bmatrix} = y_{[0,L]}^\top \, \nabla\Psi \, y_{[0,L]} = Q_{\nabla\Psi}(y)(0).$$

Now, we note that

$$Q_{\nabla\Psi}(y)(0) = Q_\Phi(y)(0) = y_{[0,L]}^\top y_{[0,L]} < 0$$

which shows that $\Psi \geqslant 0$ satisfies (9.30). This proves the theorem. □

We now return to our problem of verifying stability on the basis of output data. To this end, we give the following definition of informativity for quadratic stability.

Definition 9.9. The noisy output data $y_{[0,T]}$ are called *informative for quadratic stability* if there exists a matrix $\Psi \in \mathbb{S}^{pL}$ with $\Psi \geqslant 0$ such that the QMI (9.30) holds for all $P = \begin{bmatrix} P_0 & P_1 & \cdots & P_{L-1} \end{bmatrix}$ that satisfy the QMI (9.28), with N defined by (9.27).

Informativity for quadratic stability thus means that there exists a matrix $\Psi \in \mathbb{S}^{pL}$ such that the QDF Q_Ψ is a Lyapunov function for all systems that are consistent with the data, i.e., all systems in $\mathcal{Z}_{pL}(N)$ are stable with a common Lyapunov function.

In the sequel, our aim is to establish necessary and sufficient conditions on the data $y(0), y(1), \dots, y(T)$ to be informative in this manner. The idea is to apply the strict matrix S-lemma, Theorem A.20, to obtain such conditions in the form of feasibility of a linear matrix inequality. Note however that the QMI (9.28) is in terms of the matrix $P^\top$ whereas (9.30) is in terms of P. Therefore, immediate application of the matrix S-lemma is not possible. Below, we will resolve this issue by reformulating the QMI (9.30) in terms of the variable $P^\top$. We first formulate the following instrumental lemma.

Lemma 9.10. *Let $P(\xi) = I\xi^L + P_{L-1}\xi^{L-1} + \cdots + P_1\xi + P_0$ and, as before, let $P = \begin{bmatrix} P_0 & P_1 & \cdots & P_{L-1} \end{bmatrix}$. Define the $p(L-1) \times pL$ matrix J by*

$$J := \begin{bmatrix} 0_{p(L-1),p} & I_{p(L-1)} \end{bmatrix}. \tag{9.31}$$

Then Ψ satisfies (9.30) if and only it satisfies the (standard) Lyapunov inequality

$$\begin{bmatrix} J \\ -P \end{bmatrix}^\top \Psi \begin{bmatrix} J \\ -P \end{bmatrix} - \Psi < 0. \tag{9.32}$$

Moreover, if $\Psi \geqslant 0$ satisfies (9.30) then $\Psi > 0$.

Proof. By inspection, it can be seen that (9.30) can be reformulated as (9.32). Suppose $\Psi \geqslant 0$ satisfies (9.30). It then immediately follows that

$$\Psi \geqslant \Psi - \begin{bmatrix} J \\ -P \end{bmatrix}^\top \Psi \begin{bmatrix} J \\ -P \end{bmatrix} > 0.$$

□

Using two Schur complement arguments, the strict Lyapunov inequality (9.32) can be seen to be equivalent to

$$\Psi^{-1} - \begin{bmatrix} J \\ -P \end{bmatrix} \Psi^{-1} \begin{bmatrix} J \\ -P \end{bmatrix}^\top > 0, \quad \Psi > 0. \tag{9.33}$$

Using as an intermediate step that, obviously,

$$\begin{bmatrix} J \\ -P \end{bmatrix} = \begin{bmatrix} J \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ P \end{bmatrix}$$

it can be seen that (9.33) holds if and only if $\Psi > 0$ and

$$\begin{bmatrix} I_{pL} \\ P^\top \begin{bmatrix} 0 & -I_p \end{bmatrix} \end{bmatrix}^\top M \begin{bmatrix} I_{pL} \\ P^\top \begin{bmatrix} 0 & -I_p \end{bmatrix} \end{bmatrix} > 0 \tag{9.34}$$

where the $2pL \times 2pL$ matrix M is defined by

$$M := \begin{bmatrix} \Psi^{-1} - \begin{bmatrix} J \\ 0 \end{bmatrix} \Psi^{-1} \begin{bmatrix} J \\ 0 \end{bmatrix}^\top & -\begin{bmatrix} J \\ 0 \end{bmatrix} \Psi^{-1} \\ -\Psi^{-1} \begin{bmatrix} J \\ 0 \end{bmatrix}^\top & -\Psi^{-1} \end{bmatrix}. \tag{9.35}$$

From the above we see that informativity for quadratic stability is equivalent to the existence of $\Psi > 0$ such that the QMI (9.34) holds for all coefficient matrices $P = \begin{bmatrix} P_0 & P_1 & \cdots & P_{L-1} \end{bmatrix}$ that satisfy the QMI (9.28). In terms of solution sets of QMIs (see Section A.2), this can now be restated as

$$P^\top \in \mathcal{Z}_{pL}(N) \implies P^\top \begin{bmatrix} 0 & -I_p \end{bmatrix} \in \mathcal{Z}^+_{pL}(M)$$

or equivalently,

$$\mathcal{Z}_{pL}(N) \begin{bmatrix} 0 & -I_p \end{bmatrix} \subseteq \mathcal{Z}^+_{pL}(M). \tag{9.36}$$

In order to be able to apply the strict matrix S-lemma Theorem A.20 we want to express the (projected) set on the left in (9.36) as the solution set of a QMI. To this end, define

$$\bar{N} := \begin{bmatrix} \begin{bmatrix} 0 & -I_p \end{bmatrix} & 0 \\ 0 & I_{pL} \end{bmatrix}^\top N \begin{bmatrix} \begin{bmatrix} 0 & -I_p \end{bmatrix} & 0 \\ 0 & I_{pL} \end{bmatrix}. \tag{9.37}$$

Then, indeed, we have the following lemma.

Lemma 9.11. *Assume that the Hankel matrix $H_1(y)$ of depth L has full row rank. Then $\mathcal{Z}_{pL}(N) \begin{bmatrix} 0 & -I_p \end{bmatrix} = \mathcal{Z}_{pL}(\bar{N})$.*

Proof. Note that N is partitioned as

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

with $N_{22} = H_1(y)\Pi_{22}H_1(y)^\top$. By Assumption 9.4 we have $\Pi_{22} < 0$ and therefore $N_{22} < 0$ because $H_1(y)$ has full row rank. The true system is consistent with the data and therefore $\mathcal{Z}_{pL}(N)$ is nonempty. From (A.10), we see have that $N \,|\, N_{22} \geqslant 0$. The result then follows from Theorem A.7. □

Summarizing our findings up to now, we see that under the assumption that $H_1(y)$ has full row rank, informativity for quadratic stability is equivalent to the existence of $\Psi > 0$ such that the inclusion $\mathcal{Z}_{pL}(\bar{N}) \subseteq \mathcal{Z}^+_{pL}(M)$ holds. This inclusion is dealt with by Theorem A.20.

Lemma 9.12. *Let $\Psi > 0$ and let M be given by (9.35). Assume that $H_1(y)$ has full row rank. Then $\mathcal{Z}_{pL}(\bar{N}) \subseteq \mathcal{Z}^+_{pL}(M)$ if and only if there exists $\alpha \geqslant 0$ such that*

$$M - \alpha \bar{N} > 0. \tag{9.38}$$

Proof. We check the conditions of Theorem A.20 on $\bar{N}$. Note that

$$\bar{N} = \begin{bmatrix} \bar{N}_{11} & \bar{N}_{12} \\ \bar{N}_{21} & \bar{N}_{22} \end{bmatrix}$$

$$\bar{N}_{11} = \begin{bmatrix} 0 \\ -I_p \end{bmatrix} N_{11} \begin{bmatrix} 0 & -I_p \end{bmatrix}, \quad \bar{N}_{12} = \begin{bmatrix} 0 \\ -I_p \end{bmatrix} N_{12}.$$

We have $\bar{N}_{22} = H_1(y)\Pi_{22}H_1(y)^\top < 0$. Finally, the Schur complement $\bar{N} \,|\, \bar{N}_{22} \geqslant 0$ since $N \,|\, N_{22} \geqslant 0$. This completes the proof. ☐

Thus, informativity for quadratic stability is equivalent to the existence of a scalar $\alpha \geqslant 0$ and a matrix $\Psi > 0$ such that (9.38) holds. Note that due to the negative definite lower right block in M, the scalar α is necessarily positive. By scaling the inequality (9.38) we can therefore take $\alpha = 1$. Putting $\Phi := \Psi^{-1}$ we then finally obtain the following necessary and sufficient condition in terms of feasibility of an LMI. Recall the definition (9.31) of the matrix J.

Theorem 9.13. *Let $\bar{N}$ be given by (9.37), where N is defined by (9.27). Assume that $H_1(y)$ has full row rank. Then the output data $y_{[0,T]}$ are informative for quadratic stability if and only if there exists $\Phi \in \mathbb{S}^{pL}$ with $\Phi > 0$ such that*

$$\begin{bmatrix} \Phi - \begin{bmatrix} J \\ 0 \end{bmatrix} \Phi \begin{bmatrix} J \\ 0 \end{bmatrix}^\top & -\begin{bmatrix} J \\ 0 \end{bmatrix} \Phi \\ -\Phi \begin{bmatrix} J \\ 0 \end{bmatrix}^\top & -\Phi \end{bmatrix} - \bar{N} > 0. \tag{9.39}$$

In that case the QDF Q_Ψ with $\Psi := \Phi^{-1}$ is a Lyapunov function for all systems of the form (9.25) consistent with the data.

Remark 9.14. Note that the size of the LMI (9.39) is $2pL$ whereas the number of unknowns is $\frac{1}{2}pL(pL+1)$. These are independent of the length $T+1$ of the interval on which the input-output data are collected, and only depend on the order of the system and the number of outputs.

9.6 Data-driven stabilization of input-output AR systems

In this section we will discuss data-driven stabilization of input-output systems in AR form. We will work in the setup of Section 9.3, with systems of the form (9.12), or equivalently (9.13), with polynomial matrices as in (9.14) of given degree L. We will slightly restrict our model class and assume that the leading coefficient matrix Q_L of $Q(\xi)$ is equal to zero. In other words, we will consider systems of the form

$$P(\sigma)y = Q(\sigma)u + v \tag{9.40}$$

with

$$\begin{aligned} P(\xi) &= I\xi^L + P_{L-1}\xi^{L-1} + \cdots + P_1\xi + P_0 \\ Q(\xi) &= Q_{L-1}\xi^{L-1} + \cdots + Q_1\xi + Q_0. \end{aligned} \tag{9.41}$$

This means that $P(\xi)^{-1}Q(\xi)$ is assumed to be *strictly* proper. We assume that we have noisy input-output data $(u_{[0,T]}, y_{[0,T]})$ on the interval $[0, T]$ with $T \geqslant L$. These are samples of u and y obtained from the unknown true system

$$P_{\text{true}}(\sigma)y = Q_{\text{true}}(\sigma)u + v$$

The noise v is unknown, but its samples are assumed to satisfy Assumption 9.4. Since we have assumed that $Q_L = 0$, our model class is now parametrized by $P_0, P_1, \ldots, P_{L-1}$ and $Q_0, Q_1, \ldots, Q_{L-1}$. Again denote $R(\xi) = \begin{bmatrix} -Q(\xi) & P(\xi) \end{bmatrix}$, $q = p + m$, and collect the coefficient matrices in the $p \times qL$ matrix

$$R = \begin{bmatrix} -Q_0 & P_0 & -Q_1 & P_1 & \cdots & -Q_{L-1} & P_{L-1} \end{bmatrix} \tag{9.42}$$

Associated with the input-output data, we consider the slightly adapted Hankel matrix $H'(w)$ defined by

$$H'(w) := \begin{bmatrix} w(0) & w(1) & \cdots & w(T-L) \\ w(1) & w(2) & \cdots & w(T-L+1) \\ \vdots & \vdots & & \vdots \\ w(L-1) & w(L) & \cdots & w(T-1) \\ y(L) & y(L+1) & \cdots & y(T) \end{bmatrix}.$$

Partition

$$H'(w) = \begin{bmatrix} H_1'(w) \\ H_2'(w) \end{bmatrix}$$

where $H_1'(w)$ contains the first qL rows and $H_2'(w)$ the last p rows. Clearly, the system with coefficient matrix R is consistent with the data if and only if

$$\begin{bmatrix} I \\ R^\top \end{bmatrix}^\top N \begin{bmatrix} I \\ R^\top \end{bmatrix} \geqslant 0 \tag{9.43}$$

equivalently

$$R^\top \in \mathcal{Z}_{qL}(N)$$

where

$$N := \begin{bmatrix} I & H_2'(w) \\ 0 & H_1'(w) \end{bmatrix} \Pi \begin{bmatrix} I & H_2'(w) \\ 0 & H_1'(w) \end{bmatrix}^\top . \tag{9.44}$$

Next, we will address the stabilization problem. A feedback controller for the input-output system (9.40) with $P(\xi)$ and $Q(\xi)$ of the form (9.41) will be taken to be of the form

$$G(\sigma)u = F(\sigma)y \tag{9.45}$$

with

$$\begin{aligned} G(\xi) &= I\xi^L + G_{L-1}\xi^{L-1} + \cdots + G_1\xi + G_0 \\ F(\xi) &= F_{L-1}\xi^{L-1} + \cdots + F_1\xi + F_0. \end{aligned}$$

The leading coefficient matrix of $G(\xi)$ is assumed to be the $m \times m$ identity matrix and $G_i \in \mathbb{R}^{m\times m}$, $F_i \in \mathbb{R}^{m\times p}$ for $i = 0, 1, \ldots, L-1$. The closed loop system obtained by interconnecting the system and the controller is represented by

$$\begin{bmatrix} G(\sigma) & -F(\sigma) \\ -Q(\sigma) & P(\sigma) \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ I_p \end{bmatrix} v. \tag{9.46}$$

Since the leading coefficient matrix is the $q \times q$ identity matrix, the controlled system with noise equal to zero is autonomous. We call the controller (9.45) a stabilizing controller if the controlled system (9.46) is stable, in the sense that if $v = 0$, then all solutions u and y tend to zero as time tends to infinity.

Now define

$$C(\xi) := \begin{bmatrix} G(\xi) & -F(\xi) \end{bmatrix}$$

and recall that $w = \operatorname{col}(u, y)$. Then (9.46) can equivalently be written in kernel representation as

$$\begin{bmatrix} C(\sigma) \\ R(\sigma) \end{bmatrix} w = \begin{bmatrix} 0 \\ I_p \end{bmatrix} v. \tag{9.47}$$

Collect the coefficient matrices of $F(\xi)$ and $G(\xi)$ in the matrix C defined by

$$C := \begin{bmatrix} G_0 & -F_0 & G_1 & -F_1 & \cdots & G_{L-1} & -F_{L-1} \end{bmatrix} \tag{9.48}$$

and recall definition (9.42) of the matrix R associated likewise with $R(\xi)$. Recall that the leading coefficient matrix of $\begin{bmatrix} C(\xi)^\top & R(\xi)^\top \end{bmatrix}^\top$ is the $q \times q$ identity matrix. Furthermore, the matrix $\begin{bmatrix} C^\top & R^\top \end{bmatrix}^\top$ collects the remaining coefficient matrices. An immediate application of Theorem 9.8 then yields:

Lemma 9.15. *The controlled system* (9.47) *is stable if and only if there exists* $\Psi \in \mathbb{S}^{qL}$ *such that* $\Psi \geqslant 0$ *and*

$$\begin{bmatrix} I_{qL} \\ -C \\ -R \end{bmatrix}^\top \left(\begin{bmatrix} 0_q & 0 \\ 0 & \Psi \end{bmatrix} - \begin{bmatrix} \Psi & 0 \\ 0 & 0_q \end{bmatrix} \right) \begin{bmatrix} I_{qL} \\ -C \\ -R \end{bmatrix} < 0. \tag{9.49}$$

Moreover, if $\Psi \geqslant 0$ *satisfies* (9.49), *then* $\Psi > 0$.

This leads to the following definition of informativity.

Definition 9.16. The input-output data $(u_{[0,T]}, y_{[0,T]})$ are called *informative for quadratic stabilization* if there exist $C \in \mathbb{R}^{m \times qL}$ and $\Psi \in \mathbb{S}^{qL}$ such that $\Psi \geqslant 0$ and the QMI (9.49) holds for all R satisfying (9.43), with N defined by (9.44).

Informativity for quadratic stabilization thus means that there exists a controller $C(\sigma)w = 0$ (equivalently, $G(\sigma)u = F(\sigma)y$) and a matrix $\Psi \in \mathbb{S}^{qL}$ such that the QDF Q_Ψ is a common Lyapunov function for all closed loop systems obtained by interconnecting the controller with an arbitrary system that is consistent with the data.

Below, we will derive necessary and sufficient conditions for informativity for quadratic stabilization. Similar to Section 9.5, the QMI (9.43) is in terms of the matrix $R^\top$ whereas (9.49) is in terms of R. We will therefore first reformulate the QMI (9.49) in terms of the variable $R^\top$.

Define the $q(L-1) \times qL$ matrix J by

$$J := \begin{bmatrix} 0_{q(L-1),q} & I_{q(L-1)} \end{bmatrix}. \tag{9.50}$$

By Lemma 9.10, $\Psi \in \mathbb{S}^{qL}$, $\Psi \geqslant 0$ satisfies (9.49) if and only if $\Psi > 0$ and satisfies the strict Lyapunov inequality

$$\begin{bmatrix} J \\ -C \\ -R \end{bmatrix}^\top \Psi \begin{bmatrix} J \\ -C \\ -R \end{bmatrix} - \Psi < 0$$

which is equivalent to

$$\Psi^{-1} - \begin{bmatrix} J \\ -C \\ -R \end{bmatrix} \Psi^{-1} \begin{bmatrix} J \\ -C \\ -R \end{bmatrix}^\top > 0, \quad \Psi > 0. \tag{9.51}$$

By writing

$$\begin{bmatrix} J \\ -C \\ -R \end{bmatrix} = \begin{bmatrix} J \\ -C \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ R \end{bmatrix}$$

it can be seen that (9.51) holds if and only if $\Psi > 0$ and

$$\begin{bmatrix} I_{qL} \\ R^\top \begin{bmatrix} 0 & 0 & -I_p \end{bmatrix} \end{bmatrix}^\top M \begin{bmatrix} I_{qL} \\ R^\top \begin{bmatrix} 0 & 0 & -I_p \end{bmatrix} \end{bmatrix} > 0. \tag{9.52}$$

where the $2qL \times 2qL$ matrix M is defined by

$$M := \begin{bmatrix} \Psi^{-1} - \begin{bmatrix} J \\ -C \\ 0 \end{bmatrix} \Psi^{-1} \begin{bmatrix} J \\ -C \\ 0 \end{bmatrix}^\top & -\begin{bmatrix} J \\ -C \\ 0 \end{bmatrix} \Psi^{-1} \\ -\Psi^{-1} \begin{bmatrix} J \\ -C \\ 0 \end{bmatrix}^\top & -\Psi^{-1} \end{bmatrix}. \tag{9.53}$$

Thus we see that informativity for quadratic stabilization is equivalent to the existence of an $m \times qL$ matrix C and a matrix $\Psi \in \mathbb{S}^{qL}$, $\Psi > 0$ such that the QMI (9.52) holds for all coefficient matrices R that satisfy the QMI (9.43). The matrix C is then the coefficient matrix of a suitable controller. In terms of solution sets of QMIs this can be restated as

$$R^\top \in \mathcal{Z}_{qL}(N) \implies R^\top \begin{bmatrix} 0 & 0 & -I_p \end{bmatrix} \in \mathcal{Z}^+_{qL}(M)$$

or equivalently,

$$\mathcal{Z}_{qL}(N) \begin{bmatrix} 0 & 0 & -I_p \end{bmatrix} \subseteq \mathcal{Z}^+_{qL}(M). \tag{9.54}$$

As before, in order to be able to apply the strict matrix S-lemma in Theorem A.20, we want to express the set on the left in (9.54) as the solution set of a QMI. Define the $2qL \times 2qL$ matrix $\bar{N}$ by

$$\bar{N} := \begin{bmatrix} \begin{bmatrix} 0 & 0 & -I_p \end{bmatrix} & 0 \\ 0 & I_{qL} \end{bmatrix}^\top N \begin{bmatrix} \begin{bmatrix} 0 & 0 & -I_p \end{bmatrix} & 0 \\ 0 & I_{qL} \end{bmatrix}. \tag{9.55}$$

Then we have the following lemma, whose proof is similar to that of Lemma 9.11.

Lemma 9.17. *Assume that the Hankel matrix $H_1'(w)$ has full row rank. Then $\mathcal{Z}_{qL}(N) \begin{bmatrix} 0 & 0 & -I_p \end{bmatrix} = \mathcal{Z}_{qL}(\bar{N})$.*

From the above we see that, under the assumption that $H_1'(w)$ has full row rank, informativity for quadratic stabilization requires the existence of C and $\Psi > 0$ such that the inclusion $\mathcal{Z}_{qL}(\bar{N}) \subseteq \mathcal{Z}^+_{qL}(M)$. holds. This inclusion is dealt with by Theorem A.20.

Lemma 9.18. *Let $\Psi > 0$, $C \in \mathbb{R}^{m \times qL}$ and M be given by (9.53). Assume that $H_1'(w)$ has full row rank. Then $\mathcal{Z}_{qL}(\bar{N}) \subseteq \mathcal{Z}^+_{qL}(M)$ if and only if there exists a scalar $\alpha \geqslant 0$ such that*

$$M - \alpha \bar{N} > 0. \tag{9.56}$$

Proof. The proof is similar to that of Lemma 9.12. □

Note that the unknowns C and Ψ appear in the matrix M in a nonlinear way, and even in the form of an inverse. By putting $\Phi := \Psi^{-1}$ we can get rid of the inverse, and rewrite the condition $M - \alpha\bar{N} \geqslant 0$ as

$$\begin{bmatrix} \Phi - \begin{bmatrix} J \\ -C \\ 0 \end{bmatrix} \Phi \begin{bmatrix} J \\ -C \\ 0 \end{bmatrix}^\top & -\begin{bmatrix} J \\ -C \\ 0 \end{bmatrix} \Phi \\ -\Phi \begin{bmatrix} J \\ -C \\ 0 \end{bmatrix}^\top & -\Phi \end{bmatrix} - \alpha\bar{N} > 0. \tag{9.57}$$

Thus, informativity for quadratic stabilization holds if and only if there exists $\Phi > 0$, a matrix C, and a scalar $\alpha \geqslant 0$ such that (9.57) holds. Note that α must be positive due to the negative definite lower right block in (9.57). By scaling Φ we can therefore take $\alpha = 1$. By introducing the new variable $D := -C\Phi$ and taking a suitable Schur complement, (9.57) can then be reformulated as the following LMI in the unknowns Φ and D:

$$\begin{bmatrix} \Phi & -\begin{bmatrix} J\Phi \\ D \\ 0 \end{bmatrix} & \begin{bmatrix} J\Phi \\ D \\ 0 \end{bmatrix} \\ -\begin{bmatrix} J\Phi \\ D \\ 0 \end{bmatrix}^\top & -\Phi & 0 \\ \begin{bmatrix} J\Phi \\ D \\ 0 \end{bmatrix}^\top & 0 & \Phi \end{bmatrix} - \begin{bmatrix} \bar{N} & 0 \\ 0 & 0_{qL} \end{bmatrix} > 0. \tag{9.58}$$

This then immediately leads to the following characterization of informativity for quadratic stabilization and a method to compute a suitable feedback controller together with a common Lyapunov function.

Theorem 9.19. *Assume that $H_1'(w)$ has full row rank. Let the matrix $\bar{N}$ be given by (9.55), with N defined by (9.44). Then the input-output data $(u_{[0,T]}, y_{[0,T]})$ are informative for quadratic stabilization if and only if there exist matrices $D \in \mathbb{R}^{m\times qL}$ and $\Phi \in \mathbb{S}^{qL}$ such that $\Phi > 0$ and the LMI (9.58) holds.*

In that case, the feedback controller with coefficient matrix $C := -D\Phi^{-1}$ stabilizes all systems of the form (9.40) that are consistent with the input-output data. Moreover, the QDF Q_Ψ with $\Psi := \Phi^{-1}$ is a common Lyapunov function for all closed loop systems.

Remark 9.20. Thus, in order to compute a controller that stabilizes all systems consistent with the data and which gives a common Lyapunov function, first compute the matrix $\bar{N}$ using the Hankel matrix associated with the data. Next, check feasibility of the LMI (9.58) and, if it is feasible, compute D and Φ. An AR representation of the controller with coefficient matrix $C = -D\Phi^{-1}$ is then obtained as follows: partition $C := \begin{bmatrix} G_0 & -F_0 & G_1 & -F_1 & \cdots & G_{L-1} & -F_{L-1} \end{bmatrix}$ with $F_i \in \mathbb{R}^{m\times p}$ and $G_i \in \mathbb{R}^{m\times m}$. Next define $F(\xi) := F_{L-1}\xi^{L-1} + \cdots + F_0$ and $G(\xi) := I\xi^L + G_{L-1}\xi^{L-1} + \cdots + G_0$. The corresponding controller is then given in AR representation by $G(\sigma)u = F(\sigma)y$.

Remark 9.21. The size of the LMI (9.58) is $3qL$, while the number of unknowns is $\frac{1}{2}qL(qL + 2m + 1)$. Again these numbers do not depend on T.

9.7 Reduction of computational complexity

In this section we will again take a look at the data-driven stabilization problem. In Section 9.6 we showed that finding a controller that stabilizes all systems that are consistent with the data requires checking feasibility of the LMI (9.58). The size of this LMI is $3qL$, while the number of unknowns is $\frac{1}{2}qL(qL + 2m + 1)$, both independent of the time horizon T. The unknowns in the LMI (9.58) are the matrices Φ and D that together lead to a controller and a common Lyapunov function. In the present section we will decouple the computation of the common Lyapunov function from that of the controller. This will lead to checking feasibility of an LMI of smaller size and with a smaller number of unknowns.

In order to proceed, we will need the following lemma, whose proof follows from Theorem A.6 and Corollary A.10.

Lemma 9.22. *Let $\Pi \in \mathbf{\Pi}_{q,r}$ and let $W \in \mathbb{R}^{q\times p}$ have full column rank. Let $Y \in \mathbb{R}^{r\times p}$. Then there exists a matrix $Z \in \mathbb{R}^{r\times q}$ such that*

(a) $Z \in \mathcal{Z}_r^+(\Pi)$

(b) $ZW = Y$

if and only if $\Pi \,|\, \Pi_{22} > 0$ and $Y \in \mathcal{Z}_r^+(\Pi_W)$. If these two conditions hold and, in addition, $\Pi_{22} < 0$ then the matrix

$$Z := -\Pi_{22}^{-1}\Pi_{21} + \left(Y + \Pi_{22}^{-1}\Pi_{21}W\right)(\Pi_W \,|\, \Pi_{22})^{\dagger} W^\top (\Pi \,|\, \Pi_{22}) \tag{9.59}$$

satisfies (a) and (b).

Now consider the inequality (9.57) and recall that the existence of $\Phi > 0$ and C satisfying this inequality with $\alpha = 1$ is equivalent to informativity for quadratic stabilization. We can reformulate (9.57) as

$$\begin{bmatrix} I_{qL} & 0 \\ 0 & I_{qL} \\ \begin{bmatrix} J \\ -C \\ 0 \end{bmatrix}^\top & I_{qL} \end{bmatrix}^\top \begin{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & 0 \end{bmatrix} - \bar{N} & 0 \\ 0 & -\Phi \end{bmatrix} \begin{bmatrix} I_{qL} & 0 \\ 0 & I_{qL} \\ \begin{bmatrix} J \\ -C \\ 0 \end{bmatrix}^\top & I_{qL} \end{bmatrix} > 0. \tag{9.60}$$

Then by applying Lemma 9.22 we now obtain necessary and sufficient conditions for informativity for quadradric stabilization, together with a formula for a stabilizing controller. Define the $2qL \times (2qL - m)$ matrix W by

$$W := \begin{bmatrix} I_{q(L-1)} & 0 & 0 \\ 0 & 0_{m,p} & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_{qL} \end{bmatrix}.$$

In addition, partition the matrix $\bar{N}$ as in (9.55), where $\bar{N}_{11}$ and $\bar{N}_{22}$ are in $\mathbb{S}^{qL}$ and $\bar{N}_{12} = \bar{N}_{21}^\top \in \mathbb{R}^{qL \times qL}$.

Theorem 9.23. *Assume that $H_1'(w)$ has full row rank. Let the matrix $\bar{N}$ be given by (9.55), with N defined by (9.44). Then the input-output data $(u_{[0,T]}, y_{[0,T]})$ are informative for quadratic stabilization if and only if there exists $\Phi \in \mathbb{S}^{qL}$ such that*

$$\Phi > \bar{N} \,|\, \bar{N}_{22} \tag{9.61}$$

and

$$\begin{bmatrix} W \\ [J^\top \; 0 \; I_{qL}] \end{bmatrix}^\top \begin{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & 0 \end{bmatrix} - \bar{N} & 0 \\ 0 & -\Phi \end{bmatrix} \begin{bmatrix} W \\ [J^\top \; 0 \; I_{qL}] \end{bmatrix} > 0. \tag{9.62}$$

Moreover, if Φ satisfies these two LMIs, then the controller with coefficient matrix C defined by

$$C^\top := -\begin{bmatrix} J^\top & 0 & I_{qL} \end{bmatrix} \left(W^\top \left(\begin{bmatrix} \Phi & 0 \\ 0 & 0 \end{bmatrix} - \bar{N} \right) W \right)^{-1} W^\top \begin{bmatrix} \Phi & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0_{q(L-1),m} \\ I_m \\ 0_{(p+qL),m} \end{bmatrix} \tag{9.63}$$

satisifies (9.60). As a consequence, this controller stabilizes all systems consistent with the data, and the resulting closed loop systems have common Lyapunov function Q_Ψ with $\Psi := \Phi^{-1}$.

Proof. We first prove the 'only if' statement. Since $\Phi > 0$, it follows immediately from (9.60) that

$$\begin{bmatrix} \Phi & 0 \\ 0 & 0 \end{bmatrix} - \bar{N} > 0. \tag{9.64}$$

In turn, this implies $\Phi > \bar{N} \mid \bar{N}_{22}$. By multiplying (9.60) from the right by W and from the left by its transpose, we obtain the inequality (9.62).

To prove the converse implication, recall that $\bar{N} \mid \bar{N}_{22} \geqslant 0$. Hence it follows from (9.61) that $\Phi > 0$, and, using the fact that $\bar{N}_{22} < 0$, that (9.64) holds. From this it follows that the matrix Π defined by

$$\Pi := \begin{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & 0 \end{bmatrix} - \bar{N} & 0 \\ 0 & -\Phi \end{bmatrix}$$

is in the set $\mathbf{\Pi}_{2qL,qL}$. Then, applying Lemma 9.22 to Π, W and $Y := \begin{bmatrix} J^\top & 0 & I_{qL} \end{bmatrix}$ shows that there exists a matrix C such that (9.60) is satisfied. In other words, the data are informative for quadratic stabilization.

Finally, we will prove the formula (9.63) for $C^\top$. To this end, again apply Lemma 9.22 to Π, W and $Y = \begin{bmatrix} J^\top & 0 & I_{qL} \end{bmatrix}$. Introduce the shorthand notation

$$\Delta := \left(W^\top \left(\begin{bmatrix} \Phi & 0 \\ 0 & 0 \end{bmatrix} - \bar{N} \right) W \right)^{-1}.$$

By (9.59), a 'structured' element $\begin{bmatrix} J^\top & -C^\top & 0 & I_{qL} \end{bmatrix}$ in the set $\mathcal{Z}^+_{qL}(\Pi)$ is given by

$$\begin{bmatrix} J^\top & -C^\top & 0 & I_{qL} \end{bmatrix} = \begin{bmatrix} J^\top & 0 & I_{qL} \end{bmatrix} \Delta W^\top \left(\begin{bmatrix} \Phi & 0 \\ 0 & 0 \end{bmatrix} - \bar{N} \right)$$

As such, a controller is given by

$$C^\top = - \begin{bmatrix} J^\top & 0 & I_{qL} \end{bmatrix} \Delta W^\top \left(\begin{bmatrix} \Phi & 0 \\ 0 & 0 \end{bmatrix} - \bar{N} \right) \begin{bmatrix} 0_{q(L-1),m} \\ I_m \\ 0_{(p+qL),m} \end{bmatrix}.$$

It is easily verified that

$$\bar{N} \begin{bmatrix} 0_{q(L-1),m} \\ I_m \\ 0_{(p+qL),m} \end{bmatrix} = 0$$

Thus we conclude that $C^\top$ is given by (9.63) as claimed. □

Remark 9.24. Note that we have indeed managed to reduce the size and the number of unknowns. The total size of the LMIs (9.61) and (9.62) is equal to $3qL - m$, whereas the number of unknowns has been reduced to $\frac{1}{2}qL(qL+1)$. The computation of the controller has been decoupled from that of Φ. Indeed, a stabilizing controller is now computed using (9.63) in terms of Φ.

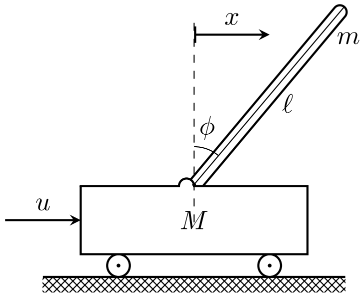


Figure 9.1: The cart and pendulum, with the parameters noted.

9.8 Simulation example

In this section we will apply the theory established in Section 9.6 to the design of a stabilizing controller for an inverted pendulum on a cart.

Consider a standard inverted pendulum on a cart as depicted in Figure 9.1. Here, m and ℓ denote the mass and length of the pendulum. The mass and coefficient of friction of the cart are denoted by M and b. Lastly, we consider the following variables: the horizontal displacement of the cart is given by x, the angle of the pendulum from the (unstable) equilibrium is ϕ, and the force applied to the cart is denoted by u.

Assuming that M, m, and ℓ are nonzero, it is straightforward to derive the following equations of motion:

$$\begin{aligned}(M+m)\ddot{x} + b\dot{x} - m\ell\ddot{\phi}\cos(\phi) + m\ell\dot{\phi}^2\sin(\phi) &= u\\ \ell\ddot{\phi} - g\sin(\phi) &= \ddot{x}\cos(\phi)\end{aligned}$$

In order to bring this model into the form used in this chapter, we discretize and then linearize it. Denoting the step size of the discretization by δ, we then obtain the linear discrete time model

$$\begin{aligned}&\begin{bmatrix} x(t+2) \\ \phi(t+2) \end{bmatrix} + \begin{bmatrix} -2+\frac{\delta b}{M} & 0 \\ \frac{\delta b}{M\ell} & -2 \end{bmatrix} \begin{bmatrix} x(t+1) \\ \phi(t+1) \end{bmatrix} \\ &+ \begin{bmatrix} 1-\frac{\delta b}{M} & -\frac{\delta^2 gm}{M} \\ -\frac{\delta b}{M\ell} & 1-\frac{\delta^2 g(M+m)}{M} \end{bmatrix} \begin{bmatrix} x(t) \\ \phi(t) \end{bmatrix} = \begin{bmatrix} \frac{\delta^2}{M} \\ \frac{\delta^2}{M\ell} \end{bmatrix} u(t)\end{aligned} \tag{9.65}$$

of order $L = 2$. We take $y = \begin{bmatrix} x & \phi \end{bmatrix}^\top$. Then after incorporating an additive noise term $v(t) = [v_1(t) \;\; v_2(t)]^\top$ in (9.65), we obtain a system of the form (9.40).

In this example we let the parameters take the following values: $M = 1$kg, $m = 0.7$kg, $b = 0.1\frac{\text{N}}{\text{m/s}}$, $g = 9.8\frac{\text{m}}{\text{s}^2}$, $l = 0.5$m, and $\delta = 0.01$s. The resulting system (9.65) with additive unknown noise term v will now be considered as the true unknown system.

9.8.1 Measurements from the linearization

In the first simulation example, we collect measurements from the noisy linearized system, i.e. the system (9.65) with additive noise. We take $T = 20$, provide 2 initial conditions, and generate random inputs from the interval $[-1, 1]$:

$$\begin{aligned} U_{[0,6]} &= \begin{bmatrix} -0.9960 & -0.7388 & -0.6322 & 0.6612 & -0.8090 & -0.2520 & 0.1023 \end{bmatrix} \\ U_{[7,13]} &= \begin{bmatrix} -0.7179 & -0.0428 & -0.8528 & 0.4309 & 0.6413 & 0.6225 & 0.2019 \end{bmatrix} \\ U_{[14,19]} &= \begin{bmatrix} 0.7475 & -0.1559 & -0.5855 & -0.7585 & -0.3562 & -0.6643 \end{bmatrix}. \end{aligned}$$

As for the matrix of noise samples V, we will assume a noise model of the form (9.16) by considering $VV^\top \leqslant \varepsilon I_2$. Note that, in order to discretize the system and make the leading coefficient equal to I_2, the dynamics were multiplied by a factor of δ^2. Indeed, it is seen in (9.65) that the effect of the input u on the dynamics is proportional to δ^2. Therefore, it is reasonable to assume that the same holds for the noise signal v. Consequently, ε can be assumed to be proportional to δ^4. In the present example, we therefore take $\varepsilon = 10^{-2}\delta^4$.

We now generate a random noise signal that satisfies the noise model and apply the initial conditions, inputs and noise to the linearized system (9.65) with additive noise. The measurements resulting from this are given as

$$\begin{aligned} Y_{[0,6]} &= \begin{bmatrix} 0.1000 & 0.1010 & 0.1020 & 0.1029 & 0.1039 & 0.1050 & 0.1061 \\ 0.1000 & 0.0990 & 0.0981 & 0.0974 & 0.0969 & 0.0969 & 0.0970 \end{bmatrix} \\ Y_{[7,13]} &= \begin{bmatrix} 0.1072 & 0.1084 & 0.1096 & 0.1108 & 0.1121 & 0.1134 & 0.1149 \\ 0.0974 & 0.0982 & 0.0991 & 0.1003 & 0.1017 & 0.1035 & 0.1058 \end{bmatrix} \\ Y_{[14,20]} &= \begin{bmatrix} 0.1165 & 0.1182 & 0.1200 & 0.1219 & 0.1238 & 0.1258 & 0.1277 \\ 0.1085 & 0.1116 & 0.1153 & 0.1192 & 0.1235 & 0.1279 & 0.1327 \end{bmatrix}. \end{aligned}$$

We will use Theorem 9.19 to show that these measurements are informative for quadratic stabilization. For this, we first form the matrices H_1', H_2' and $\bar{N}$. It is straightforward to see that H_1' has full row rank. We now use Yalmip with Mosek as a solver in order to find matrices $D \in \mathbb{R}^{1\times 6}$, and $\Phi \in \mathbb{S}^6$, such that $\Phi > 0$ and the LMI (9.58) holds. Indeed, such matrices exist, and therefore the data are informative for quadratic stabilization. We can find a stabilizing controller by taking $C = -D\Phi^{-1}$, which results in

$$C = \begin{bmatrix} 0.76 & 29168.72 & -18360.21 & 0.68 & -29515.03 & 19264.40 \end{bmatrix}.$$

This corresponds to the controller of the form (9.45) given by

$$\begin{aligned} u(t+2) + 0.68u(t+1) + 0.76u(t) &= 29515.03x(t+1) \\ &- 19264.40\phi(t+1) - 29168.72x(t) + 18360.21\phi(t). \end{aligned}$$

The large difference in magnitude of the gains corresponding to x and ϕ and those corresponding to u is caused by the discretization.

In Figure 9.2 we can see the results of applying this controller to the linear discretized model, with noise $v = 0$. To be precise, we plot both x in Figure 9.2(a) and ϕ in Figure 9.2(b) for 200 steps originating from a given initial condition. This illustrates that the controller stabilizes the linearized system, as was guaranteed by Theorem 9.19.

9.8.2 Measurements from the nonlinear system

In this example, instead of measuring the linear system (9.65) with a bounded noise term, we will perform measurements on the (discretized) nonlinear system directly. This means that we interpret the noise term $v(t)$ of the linear system as the effect of the nonlinearities. Again, we provide 2 initial conditions and take $T = 20$. We will generate measurements close to the equilibrium, in order to keep the effect of the nonlinearities relatively small. As such, we will assume that $VV^\top \leqslant 10^{-4}\delta^4 I_2$, which we will validate experimentally.

Again, we generate random inputs from the interval $[-1, 1]$, which results in:

$$\begin{aligned} U_{[0,6]} &= \begin{bmatrix} -0.6358 & -0.2516 & 0.6150 & -0.1941 & -0.4534 & -0.8523 & 0.1926 \end{bmatrix} \\ U_{[7,13]} &= \begin{bmatrix} -0.6554 & -0.0237 & 0.3687 & -0.0440 & -0.2577 & 0.3739 & 0.5910 \end{bmatrix} \\ U_{[14,19]} &= \begin{bmatrix} -0.1870 & 0.2488 & -0.6610 & 0.7050 & -0.3602 & 0.1016 \end{bmatrix}. \end{aligned}$$

If we apply these to the nonlinear system with the given initial conditions, we obtain the following measurements.

$$\begin{aligned} Y_{[0,6]} &= \begin{bmatrix} 0.1000 & 0.1010 & 0.1020 & 0.1029 & 0.1040 & 0.1050 & 0.1061 \\ 0.0400 & 0.0390 & 0.0380 & 0.0371 & 0.0364 & 0.0358 & 0.0353 \end{bmatrix} \\ Y_{[7,13]} &= \begin{bmatrix} 0.1070 & 0.1081 & 0.1090 & 0.1100 & 0.1111 & 0.1121 & 0.1132 \\ 0.0347 & 0.0342 & 0.0337 & 0.0333 & 0.0332 & 0.0331 & 0.0330 \end{bmatrix} \\ Y_{[14,20]} &= \begin{bmatrix} 0.1143 & 0.1155 & 0.1167 & 0.1180 & 0.1192 & 0.1205 & 0.1218 \\ 0.0332 & 0.0336 & 0.0340 & 0.0346 & 0.0352 & 0.0361 & 0.0370 \end{bmatrix}. \end{aligned}$$

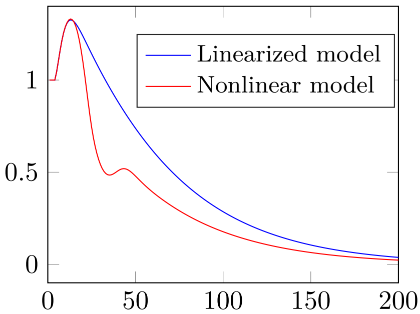
For the sake of simulations, we note that the effects of the nonlinearities for these initial conditions and inputs, as captured in the matrix V, indeed satisfy the assumed noise model. Similar to earlier, we note that H_1' has full row rank, and we can find Φ and D such that (9.58) holds. This means that the data are informative for quadratic stabilization. By taking $C = -D\Phi^{-1}$, we obtain

$$C = \begin{bmatrix} 1.03 & 27778.78 & -19129.66 & 0.85 & -27967.57 & 20120.40 \end{bmatrix}.$$

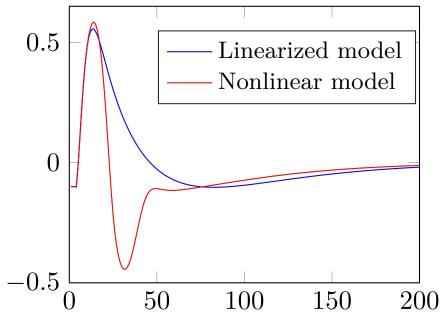
This corresponds to a controller given by:

$$\begin{aligned} u(t+2) + 0.85u(t+1) + 1.03u(t) &= 27967.57x(t+1) \\ &- 20120.40\phi(t+1) - 27778.78x(t) + 19129.66\phi(t). \end{aligned}$$

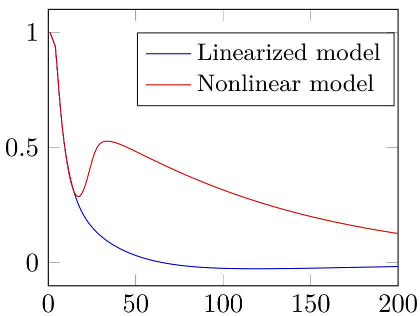
As before, we apply the resulting controller to both the discretization of the nonlinear model and its linearization (9.65) without noise. For both models and a given initial condition the values of the position of the cart for 200 steps are shown in Figure 9.2(c). In Figure 9.2(d) we show the corresponding angles of the pendulum for the same interval of time.



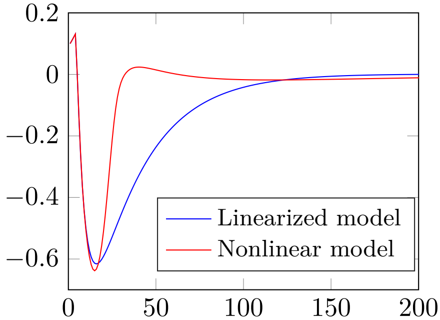
(a) The position x of the cart for the situation of Section 9.8.1.



(b) The angle ϕ of the pendulum for the situation of Section 9.8.1.



(c) The position x of the cart for the situation of Section 9.8.2.



(d) The angle ϕ of the pendulum for the situation of Section 9.8.2.

Figure 9.2: The results of interconnecting the controllers of Section 9.8.1 (top) and Section 9.8.2 (bottom) to the linearized and nonlinear model.

9.9 Notes and references

The results of this chapter are based on the paper [174]. They rely heavily on ideas from the behavioral approach to systems and control [186–189]. For a textbook reference on this subject we refer to [129]. In particular, one of the central objects of study has been the concept of quadratic difference forms (QDFs).

Quadratic differential forms were first introduced in continuous time by Willems and Trentelman in [191]. There, they were considered as a vehicle for constructing, among others, Lyapunov functions for linear systems described by higher order differential equations. In fact, one of the main ideas of that paper is to characterize stability of autonomous linear systems in autoregressive form by means of a QDF Lyapunov function that is a quadratic function of the external variables of the system and their derivatives up to a certain order. The topic of controller synthesis using quadratic differential forms has been studied in the two-part paper [192, 193], leading to the solution to the $\mathcal{H}_\infty$ control problem in a behavioral setting.

In discrete-time, quadratic difference forms have been studied in the literature. In particular, we refer to the papers [90, 91] that treat stability analysis of systems described by higher order difference equations. In this case, QDF Lyapunov functions are quadratic functions of a number of shifts of the external variables of the system.

In the context of data-driven analysis, QDFs have been used before to assess dissipativity properties of linear systems from data. In particular, we point towards the paper [107] and the more recent work [140].

The problem of stabilization of input-output systems using noisy input-output data has been considered in the papers [20, 44, 157]. A general strategy, adopted by all these papers, is to rely on an auxiliary state-space representation of the system with a state comprised of shifts of the inputs and outputs. This leads to an input-state-output system to which techniques for state data are applicable. A potential downside of this approach is that the obtained state-space systems are structured in the sense that the involved matrices contain both known and unknown block entries. If this structure is not taken fully into account, this may lead to rather conservative conditions for data-driven control design. Exploiting this prior knowledge on the system matrices is an important problem, which has recently been studied in [20]. The approach in Section 9.6 provides an interesting alternative to the above paper since it avoids the use of state-space representations. In fact, the idea of this chapter has been to work directly with Lyapunov functions that are functions of (external) input and output variables, thereby obviating the need for state-space respresentations. Nonetheless, we do point out that the linear matrix inequality condition for the existence of a QDF Lyapunov function in Theorem 9.8 can be interpreted as a 'standard' Lyapunov inequality (9.32) for a state-space representation of the AR system under consideration. Therefore, the problem of data-driven stabilization using input-output data could alternatively be solved using state-space representations. However, an important aspect of our approach is to take into account the structure of such state-space representations by using Lemmas 9.11 and 9.17.

10

Data-driven tracking and regulation

10.1 Introduction

This chapter deals with the classical control design problem of tracking and regulation. We will study this problem from a data-driven perspective, using the concept of informativity.

Before embarking on the data-driven approach, we will first briefly review the main idea behind this problem. Roughly speaking, the problem of tracking and regulation is the combination of two different control problems, namely the *tracking problem* and the *output regulation problem*, into one single problem.

The tracking problem is the problem of finding a feedback controller such that the output of the controlled system tracks (i.e., converges to) some a priori given reference signal. Many relevant reference signals (such as step functions, ramps, sinusoids) can be generated as solutions of a suitable autonomous linear system. Given a desired reference signal, one first constructs a suitable generating autonomous system (called the exosystem). Next, this exosystem is interconnected to the control system (called the endosystem), and a new output is defined as the difference between the original system output and the reference signal. The tracking problem is then to design a feedback controller such that the output of the interconnection converges to zero as time runs off to infinity.

On the other hand, in the output regulation problem we have a control system (again called endosystem) subjected to external disturbances, and we want to design a feedback controller such that the output of the controlled system converges to zero for any disturbance entering the system and for any initial state of the system. In the output regulation problem, a distinguishing feature is that we assume that the disturbance inputs are generated by some autonomous linear system, again called an exosystem. The output regulation problem is then to design a feedback controller for the interconnection of endo- and exosystem such that its output converges to zero for all initial states. Of course, the above two feedback design problems can be combined into the single problem of designing a controller such that the output of the controlled system tracks a given reference signal, regardless of the disturbance input entering the system, and the initial state. This problem is referred to as the problem of *tracking and regulation*, also called the *regulator problem*.

In this chapter we will study this problem in a data-driven context. It wil be assumed that the 'true' endosystem is unknown, and that we only have data on the input, endosystem state, and exosystem state in the form of samples on a finite time-interval. The exosystem is assumed to be known, since this system models the reference signals and possible disturbance inputs. Also the matrices in the output equations are assumed to be known, since these specify the design specification (namely the output that should converge to zero).

10.2 An illustrative example

We will first illustrate the problem that will be considered in this chapter by means of a simple extended example.

Example 10.1. Consider the scalar linear time-invariant system

$$x(t+1) = a_{\text{true}}x(t) + b_{\text{true}}u(t) + d(t) \tag{10.1}$$

where x is the state, u the control input, and d a disturbance input. The values of a_{true} and b_{true} in this system representation are unknown. We assume that the disturbance can be any constant signal of finite amplitude. Suppose that we want the state $x(t)$ to track the given reference signal $r(t) = \cos \frac{\pi}{2}t$, for any constant disturbance input, regardless of the initial state of the system. The problem is to design a control law for (10.1) that achieves this specification. We assume that r, x and d are available for feedback and allow control laws of the form

$$u(t) = k_1 r(t) + k_2 r(t+1) + k_3 d(t) + k_4 x(t). \tag{10.2}$$

Interconnecting (10.1) and (10.2) results in the controlled system

$$x(t+1) = (a_{\text{true}} + b_{\text{true}}k_4)x(t) + (b_{\text{true}}k_3 + 1)d(t) + b_{\text{true}}k_1 r(t) + b_{\text{true}}k_2 r(t+1)$$

where the gains k_i should be designed such that $x(t) - r(t) \to 0$ as $t \to \infty$ for any constant disturbance input d and initial state $x(0)$. It is also required that the controlled system is internally stable, in the sense that $a_{\text{true}} + b_{\text{true}}k_4$ is stable[1].

The values of a_{true} and b_{true} that represent the true system are unknown, but in the data-driven context it is assumed that we do have access to certain data. In particular, it is assumed that we have finite sequences of samples of $x(t)$, $u(t)$ and $d(t)$ on a given time interval $[0, T]$, given by

$$U_- := U_{[0,T-1]} \tag{10.3a}$$

$$X := X_{[0,T]} \tag{10.3b}$$

$$D_- := D_{[0,T-1]} \tag{10.3c}$$

[1] In this scalar context this means that it should lie in the interval $(-1, 1)$.

where, in this particular example, by assumption $d(t) = d(0)$ for $t \in [0, T-1]$. Define

$$X_+ := X_{[1,T]}$$
$$X_- := X_{[0,T-1]}.$$

It is assumed that these data are generated by the true system, so we must have $X_+ = a_{\text{true}} X_- + b_{\text{true}} U_- + D_-$. For this example, the problem of data-driven control design is now to use the data (10.3) to determine whether a suitable controller (10.2) exists, and to compute the associated gains k_1, k_2, k_3 and k_4 using only these data.

Note that in the above, both the reference signal and the disturbance signals are solutions of the autonomous linear system

$$\begin{bmatrix} r_1(t+1) \\ r_2(t+1) \\ d(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \\ d(t) \end{bmatrix} \tag{10.4}$$

with initial state $r_1(0) = 1$ and $r_2(0) = 0$, and $d(0)$ arbitrary. Indeed, it can be seen that the reference signal $r(t) = \cos \frac{\pi}{2} t$ is equal to $r_1(t)$. In addition, the solutions $d(t)$ are all constant signals of finite amplitude. The autonomous system (10.4) is called the *exosystem*.

The interconnection of the (unknown) to be controlled system (10.1) (called the *endosystem*) with the exosystem (10.4), is represented by

$$\begin{bmatrix} r_1(t+1) \\ r_2(t+1) \\ d(t+1) \\ x(t+1) \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & a_{\text{true}} \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \\ d(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_{\text{true}} \end{bmatrix} u(t). \tag{10.5}$$

In this representation, the part corresponding to the exosystem is known, but the part corresponding to the endosystem (specifically: a_{true} and b_{true}) is unknown. We now also specify a (known) output equation

$$z(t) = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \\ d(t) \\ x(t) \end{bmatrix}.$$

Then the problem of our example can be rephrased as: using only the data (10.3), design a full state feedback control law

$$u(t) = k_1 r_1(t) + k_2 r_2(t) + k_3 d(t) + k_4 x(t)$$

for the system (10.5) such that in the controlled system we have $z(t) \to 0$ as $t \to \infty$ for the initial states $r_1(0) = 1$, $r_2(0) = 0$, and $d(0)$ arbitrary, while internal stability is achieved in the sense that $a_{\text{true}} + b_{\text{true}}k_4$ is stable. In order to allow tracking of signals from the richer class of all reference signals of the form $r(t) = A\cos(\frac{1}{2}\pi t + \omega)$ (A and ω are determined by the initial states $r_1(0) = 1$ and $r_2(0)$), we may slightly relax the problem formulation and require $z(t) \to 0$ as $t \to \infty$ for *all* initial states $r_1(0), r_2(0)$ and $d(0)$. ■

10.3 Informativity for regulator design

In this section we will formulate the problem illustrated in the example of the previous section in a general framework. To this end, consider the model class of endosystems of the form

$$x_2(t+1) = A_2 x(t) + B_2 u(t) + A_3 x_1(t). \tag{10.6}$$

Here, x_2 is the n_2-dimensional state, u the m-dimensional input, and x_1 the n_1-dimensional state of the exosystem

$$x_1(t+1) = A_1 x_1(t) \tag{10.7}$$

that generates all possible reference signals and disturbance inputs. The matrices A_2 and B_2 will be assumed to be unknown, but the matrix A_1 is assumed to be known. Also A_3 is assumed to be a known matrix that represents how the endosystem is interconnected with the exosystem. Later on in this chapter, in Section 10.7, we will treat the case that, in addition to A_2 and B_2, also the coupling matrix A_3 is unknown. The output to be regulated is specified by

$$z(t) = D_1 x_1(t) + D_2 x_2(t) + E u(t) \tag{10.8}$$

where also D_1, D_2 and E are assumed to be known. Accordingly, our model class $\mathcal{M}$ consists of all systems given by (10.6), (10.7) and (10.8), parametrized by the matrices A_2 and B_2. The true (unknown) values of these matrices are given by $A_{2,\text{true}}$ and $B_{2,\text{true}}$.

By interconnecting the endosystem (10.6) with the state feedback controller

$$u(t) = K_1 x_1(t) + K_2 x_2(t) \tag{10.9}$$

we obtain the controlled system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ A_3 + B_2 K_1 & A_2 + B_2 K_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{10.10a}$$

$$z(t) = \begin{bmatrix} D_1 + EK_1 & D_2 + EK_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{10.10b}$$

If $z(t) \to 0$ as $t \to \infty$ for all initial states $x_1(0)$ and $x_2(0)$, we say that the controlled system is *output regulated.* If $A_2 + B_2K_2$ is a stable matrix we call the controlled system *endo-stable.* If the control law (10.9) makes the controlled system both output regulated and endo-stable, we call it a *regulator.*

Since we do not know the true values $A_{2,\text{true}}$ and $B_{2,\text{true}}$ representing the endosystem (10.6), the design of a regulator can only be based on available data. We will assume that the data are finite sequences of samples of $x_1(t), x_2(t)$ and $u(t)$ on the time interval $[0, T]$ given by

$$\begin{aligned} U_- &:= U_{[0,T-1]} \\ X_{1-} &:= X_{1\,[0,T-1]} \\ X_2 &:= X_{2\,[0,T]}. \end{aligned}$$

The true endosystem generates these data, and therefore we must have

$$X_{2+} = A_{2,\text{true}}X_{2-} + B_{2,\text{true}}U_- + A_3X_{1-} \tag{10.12}$$

where, as before, we denote

$$\begin{aligned} X_{2-} &:= X_{2\,[0,T-1]} \\ X_{2+} &:= X_{2\,[1,T]}. \end{aligned}$$

An endosystem with system matrices (A_2, B_2) is called consistent with these data if also A_2 and B_2 satisfy the equation

$$X_{2+} = A_2X_{2-} + B_2U_- + A_3X_{1-}\ . \tag{10.13}$$

The set of all (A_2, B_2) that are consistent with the data is again denoted by $\Sigma_{\mathcal{D}}$, i.e.,

$$\Sigma_{\mathcal{D}} := \{(A_2, B_2) \mid (10.13) \text{ holds}\}\,. \tag{10.14}$$

Since (10.12) is assumed to hold, the true endosystem $(A_{2,\text{true}}, B_{2,\text{true}})$ is in $\Sigma_{\mathcal{D}}$. In general, the equation (10.13) does not specify the true system uniquely, and many endosystems (A_2, B_2) may be consistent with the same data.

Now we turn to controller design based on the data (U_-, X_{1-}, X_2). Note that, since on the basis of the given data we cannot distinguish between the true endosystem and any other endosystem consistent with these data, a suitable data-based regulator for the true system should be a regulator for any system (A_2, B_2) in $\Sigma_{\mathcal{D}}$. If such regulator exists, we call the data *informative for regulator design.* More precisely:

Definition 10.2. We say that the data (U_-, X_{1-}, X_2) are *informative for regulator design* if there exists K_1 and K_2 such that the control law $u(t) = K_1x_1(t) + K_2x_2(t)$ is a regulator for any endosystem such that (A_2, B_2) in $\Sigma_{\mathcal{D}}$.

The problem is to find conditions on the data (U_-, X_{1-}, X_2) to be informative for regulator design. In addition, in case that these conditions are satisfied we would like to compute a regulator using only these data. Before addressing this problem, in the next section we will review some basic material on the regulator problem.

10.4 Some background on the regulator problem

In order to be able to proceed with our data-driven approach, in this section we briefly review some basic material on the regulator problem. We will distinguish between analysis and design.

We first consider the analysis question under what conditions a controlled system is endo-stable and output regulated. Consider a given autonomous linear system represented by

$$\begin{aligned} x_1(t+1) &= A_1 x_1(t) \\ x_2(t+1) &= A_2 x_2(t) + A_3 x_1(t) \\ z(t) &= D_1 x_1(t) + D_2 x_2(t). \end{aligned} \tag{10.15}$$

In accordance with the terminology introduced in Section 2.2, we call this system endo-stable if A_2 is a stable matrix. We call it output regulated if $z(t) \to 0$ as $t \to \infty$ for all initial states $x_1(0)$ and $x_2(0)$. The following proposition gives conditions under which (10.15) is endo-stable and output regulated.

Proposition 10.3. *Assume that A_1 is anti-stable*[2]*. Then the system (10.15) is endo-stable and output regulated if and only if A_2 is stable and there exists a matrix S satisfying the equations*

$$SA_1 - A_2 S = A_3 \tag{10.16a}$$

$$D_1 + D_2 S = 0. \tag{10.16b}$$

In this case, S is unique.

Proof. Assume that A_2 is stable and that S is a solution to the equations (10.16). Then, by definition, (10.15) is endo-stable. We will prove it is output regulated. Define $v(t) := x_2(t) - S x_1(t)$. Then

$$v(t+1) = A_2 v(t) + (A_2 S - S A_1 + A_3) x_1(t) \tag{10.17a}$$

$$z(t) = D_2 v(t) + (D_1 + D_2 S) x_1(t). \tag{10.17b}$$

[2]We say that a square matrix is *anti-stable* if all its eigenvalues λ satisfy $|\lambda| \geqslant 1$.

As a consequence,

$$v(t+1) = A_2 v(t) \tag{10.18a}$$
$$z(t) = D_2 v(t). \tag{10.18b}$$

Since A_2 is stable, it follows that $z(t) \to 0$ as $t \to \infty$, so (10.15) is output regulated.

For the converse, assume that (10.15) is endo-stable and output regulated. Then A_2 is stable. Since A_1 is anti-stable, the Sylvester equation (10.16a) has a (unique) solution S (see [160, Thm. 9.6]). We will show that it satisfies (10.16b). Indeed, $v(t)$ satisfies (10.18a) and $z(t)$ satisfies (10.17b). Since the system is output regulated and A_2 is stable, we must have $(D_1 + D_2 S)x_1(t) \to 0$ as $t \to \infty$. Since A_1 is anti-stable, this immediately implies that $D_1 + D_2 S = 0$. □

Next, we consider the design problem and review conditions under which, for a given interconnection of an endosystem and exosystem, there exists a regulator, i.e., a controller that makes the controlled system endo-stable and output regulated. Indeed, given the to-be-controlled system

$$\begin{aligned} x_1(t+1) &= A_1 x_1(t) \\ x_2(t+1) &= A_2 x_2(t) + B_2 u(t) + A_3 x_1(t) \\ z(t) &= D_1 x_1(t) + D_2 x_2(t) + E u(t) \end{aligned}$$

the following proposition gives conditions for the existence of a regulator, and formulas to compute one.

Proposition 10.4. *Assume that A_1 is anti-stable. There exists a regulator of the form (10.9) if and only if (A_2, B_2) is stabilizable and there exist matrices S and V satisfying the regulator equations*

$$SA_1 - A_2 S - B_2 V = A_3, \;\; D_1 + D_2 S + EV = 0. \tag{10.19}$$

In this case, a regulator is obtained as follows: choose any K_2 such that $A_2 + B_2 K_2$ is stable, and define $K_1 := -K_2 S + V$.

Proof. Assume that a regulator (10.9) exists. Then the controlled system (10.10) is endo-stable and output regulated, so by Proposition 10.3 the following equations have a (unique) solution S:

$$SA_1 - (A_2 + B_2 K_2)S = A_3 + B_2 K_1 \tag{10.20a}$$
$$D_1 + EK_1 + (D_2 + EK_2)S = 0. \tag{10.20b}$$

Clearly, (A_2, B_2) is stabilizable. By defining $V := K_1 + K_2 S$, we also see that S and V satisfy the regulator equations (10.19). Conversely, assume that S and

V satisfy (10.19) and that (A_2, B_2) is stabilizable. Choose any K_2 such that $A_2 + B_2K_2$ is stable, and define $K_1 := -K_2S + V$. Then the equations (10.20) hold, so again by Proposition 10.3, the controlled system (10.10) is endo-stable and output regulated. □

10.5 Regulator design from data

In this section we will give necessary and sufficient conditions on the data to be informative for regulator design, and data-based formulas to compute suitable regulators. Before doing this, we first introduce the notion of *informativity for endo-stabilization.*

Definition 10.5. The data (U_-, X_{1-}, X_2) are *informative for endo-stabilization* if there exists K_2 such that $A_2 + B_2K_2$ is a stable matrix for all (A_2, B_2) in $\Sigma_{\mathcal{D}}$.

Note that a necessary condition for the data (U_-, X_{1-}, X_2) to be informative for regulator design is that they are informative for endo-stabilization. In order to obtain necessary and sufficient conditions for informativity for endo-stabilization we formulate:

Proposition 10.6. *Let T be a positive integer. Let Z, X be real $n \times T$ matrices and let U be a real $m \times T$ matrix. Consider the set*

$$\Sigma_{(Z,X,U)} := \{(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \mid Z = AX + BU\}.$$

Assume that $\Sigma_{(Z,X,U)}$ is non-empty. Then the following hold:

(a) *There exists a matrix K such that $A + BK$ is stable for all $(A, B) \in \Sigma_{(Z,X,U)}$ if and only if X has full row rank, and there exists a right-inverse $X^\sharp$ such that $ZX^\sharp$ is stable. In that case, by taking $K := UX^\sharp$ we have $A + BK$ is stable for all $(A, B) \in \Sigma_{(Z,X,U)}$.*

(b) *For any K such that $A+BK$ is stable for all $(A, B) \in \Sigma_{(Z,X,U)}$ there exists a right-inverse $X^\sharp$ such that $K = UX^\sharp$, and, moreover, $A + BK = ZX^\sharp$ for all $(A, B) \in \Sigma_{(Z,X,U)}$.*

Proof. A proof of this proposition can be given by slightly modifying the proof of Theorem 6.4. □

This immediately gives the following conditions for informativity for endo-stabilization.

Lemma 10.7. *The data (U_-, X_{1-}, X_2) are informative for endo-stabilization if and only if X_{2-} has full row rank and there exists a right inverse $X_{2-}^\sharp$ of X_{2-} such that $(X_{2+} - A_3X_{1-})X_{2-}^\sharp$ is stable. In that case, by taking $K_2 := U_-X_{2-}^\sharp$ we have $A_2 + B_2K_2$ is stable for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$.*

The following theorem now gives necessary and sufficient conditions on the data to be informative for regulator design, and explains how suitable regulators are computed using only these data.

Theorem 10.8. *Assume that A_1 is anti-stable and suppose, for simplicity, that it is diagonalizable. Then the data (U_-, X_{1-}, X_2) are informative for regulator design if and only if at least one of the following two conditions hold*

(a) *The inclusion $\operatorname{im} D_1 \subseteq \operatorname{im} E$ holds, and X_{2-} has full row rank and has a right-inverse $X_{2-}^\sharp$ such that the matrix $(X_{2+} - A_3X_{1-})X_{2-}^\sharp$ is stable and $D_2 + EU_-X_{2-}^\sharp = 0$. In this case, a regulator is found as follows: choose K_1 such that $D_1 + EK_1 = 0$ and define $K_2 := U_-X_{2-}^\sharp$.*

(b) *X_{2-} is right-invertible and it has a right-inverse $X_{2-}^\sharp$ such that the matrix $(X_{2+} - A_3X_{1-})X_{2-}^\sharp$ is stable. Moreover, there exists a solution W to the linear equations*

$$X_{2-}WA_1 - (X_{2+} - A_3X_{1-})W = A_3 \qquad (10.21a)$$

$$D_1 + (D_2X_{2-} + EU_-)W = 0. \qquad (10.21b)$$

In this case, a regulator is found as follows: take $K_1 := U_-(I - X_{2-}^\sharp X_{2-})W$ and $K_2 := U_-X_{2-}^\sharp$.

Before turning to the proof, we will explain how to apply this theorem. What we know about the system are the matrices A_1, A_3, D_1, D_2 and E, and the data (U_-, X_{1-}, X_2). The aim is to use this knowledge to compute a *single* regulator (K_1, K_2) that works *for all* endosystems (A_2, B_2) in Σ_D. Theorem 10.8 states that such regulator exists if and only if at least one of the two conditions (a) or (b) holds. If condition (a) holds then such regulator is computed as follows: choose K_1 such that $D_1 + EK_1 = 0$ and define $K_2 := U_-X_{2-}^\sharp$. If (b) holds then a regulator is computed as follows: choose $K_1 := U_-(I - X_{2-}^\sharp X_{2-})W$ and $K_2 := U_-X_{2-}^\sharp$.

Proof. We first prove sufficiency. Assume that the condition (a) holds. Since $(X_{2+} - A_3X_{1-})X_{2-}^\sharp$ is stable, the data are informative for endo-stabilization and by taking $K_2 := U_-X_{2-}^\sharp$ we have $A_2 + B_2K_2$ is stable for all $(A_2, B_2) \in \Sigma_\mathcal{D}$. Since A_1 is assumed to be anti-stable, this implies that for all $(A_2, B_2) \in \Sigma_\mathcal{D}$ there exists a unique solution S to the Sylvester equation $SA_1 - (A_2 + B_2K_2)S = A_3 + B_2K_1$. By the fact that $D_1 + EK_1 = 0$ and $D_2 + EK_2 = 0$, this solution S also satisfies $D_1 + EK_1 + (D_2 + EK_2)S = 0$. Thus, for all $(A_2, B_2) \in \Sigma_\mathcal{D}$, there exists a matrix S that satisfies the equations (10.16). It follows from

Proposition 10.3 that for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$ the controlled system is endo-stable and output regulated.

Next, assume that condition (b) holds. By Lemma 10.7, the data are informative for endo-stabilization and by taking $K_2 := U_1 X_-^\sharp$ we have $A_2 + B_2 K_2$ stable for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$. Let W satisfy the equations (10.21). Define $S := X_{2-} W$ and $V := U_- W$. Then the pair (S, V) satisfies the regulator equations (10.19) for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$. Then, by Proposition 10.4, for each such (A_2, B_2) a regulator is given by the pair (K_1, K_2), with $K_1 = -K_2 S + V = -K_2 X_{2-} W + U_- W = U_-(I - X_{2-}^\sharp X_{2-}) W$. This completes the proof of the sufficiency part.

We will now turn to the necessity part. Assume that the data are informative for regulator design. By Proposition 10.3, there exist K_1 and K_2 and for any $(A_2, B_2) \in \Sigma_{\mathcal{D}}$ a matrix $S_{(A_2,B_2)}$ such that $A_2 + B_2 K_2$ is stable and

$$\begin{aligned} S_{(A_2,B_2)} A_1 - (A_2 + B_2 K_2) S_{(A_2,B_2)} &= A_3 + B_2 K_1 \\ D_1 + E K_1 + (D_2 + E K_2) S_{(A_2,B_2)} &= 0. \end{aligned}$$

We emphasize that $S_{(A_2,B_2)}$ may depend on the choice of $(A_2, B_2) \in \Sigma_{\mathcal{D}}$. However, since $A_2 + B_2 K_2$ is stable for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$, by Proposition 10.6 there exists a right-inverse $X_{2-}^\sharp$ of X_{2-} such that $A_2 + B_2 K_2 = (X_{2+} - A_3 X_{1-}) X_{2-}^\sharp$ for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$. The latter matrix is independent of (A_2, B_2). Call it M.

Define

$$\Sigma_{\mathcal{D}}^0 := \{(A_2^0, B_2^0) \mid \begin{bmatrix} A_2^0 & B_2^0 \end{bmatrix} \begin{bmatrix} X_{2-} \\ U_- \end{bmatrix} = 0\}.$$

Note that $\Sigma_{\mathcal{D}}^0$ is the solution space of the homogeneous version of the defining equation (10.14) for $\Sigma_{\mathcal{D}}$. We now distinguish two cases, namely (i) $B_2^0 K_1 = 0$ for all $(A_2^0, B_2^0) \in \Sigma_{\mathcal{D}}^0$, and (ii) $B_2^0 K_1 \neq 0$ for some $(A_2^0, B_2^0) \in \Sigma_{\mathcal{D}}^0$.

First consider case (i). Then for all $(A_2, B_2), (\bar{A}_2, \bar{B}_2) \in \Sigma_{\mathcal{D}}$ we have $B_2 K_1 = \bar{B}_2 K_1$. Thus, there exists a *common* matrix S that solves the equations

$$\begin{aligned} S A_1 - M S &= A_3 + B_2 K_1 \\ D_1 + E K_1 + (D_2 + E K_2) S &= 0 \end{aligned}$$

for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$. From this, we obtain

$$S A_1 - \begin{bmatrix} A_2 & B_2 \end{bmatrix} \begin{bmatrix} S \\ K_2 S + K_1 \end{bmatrix} = A_3$$

for all $(A_2, B_2) \in \Sigma_{\mathcal{D}}$, and therefore

$$\begin{bmatrix} A_2^0 & B_2^0 \end{bmatrix} \begin{bmatrix} S \\ K_2 S + K_1 \end{bmatrix} = 0$$

for all $(A_2^0, B_2^0) \in \Sigma_{\mathcal{D}}^0$. This implies

$$\operatorname{im} \begin{bmatrix} S \\ K_2 S + K_1 \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} X_{2-} \\ U_- \end{bmatrix}.$$

As a consequence, there exists a matrix W such that

$$\begin{bmatrix} S \\ K_2 S + K_1 \end{bmatrix} = \begin{bmatrix} X_{2-} \\ U_- \end{bmatrix} W.$$

Clearly, W satisfies the equations (10.21), showing that condition (b) holds.

Next, consider case (ii). Let Q be a real $(n_2 + m) \times r$ matrix such that

$$\ker \begin{bmatrix} X_{2-} \\ U_- \end{bmatrix}^\top = \operatorname{im} Q.$$

Partition $Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$. Then $(A_2^0, B_2^0) \in \Sigma_{\mathcal{D}}^0$ if and only if $A_2^0 = NQ_1^\top$ and $B_2^0 = NQ_2^\top$ for some $n_2 \times r$ matrix N. Note that, by hypothesis, $Q_2^\top K_1 \neq 0$.

Let $(A_2, B_2) \in \Sigma_{\mathcal{D}}$. Recall that for any such (A_2, B_2) there exists a unique $S_{(A_2,B_2)}$ such that

$$\begin{aligned} &S_{(A_2,B_2)} A_1 - M S_{(A_2,B_2)} = A_3 + B_2 K_1 \\ &D_1 + EK_1 + (D_2 + EK_2) S_{(A_2,B_2)} = 0. \end{aligned} \tag{10.22}$$

Now let N be any real $n_2 \times r$ matrix. Then also $(A_2 + NQ_1^\top, B_2 + NQ_2^\top) \in \Sigma_{\mathcal{D}}$. Define

$$S_N := S_{(A_2,B_2)} - S_{(A_2+NQ_1^\top, B_2+NQ_2^\top)}.$$

Then clearly S_N is the unique solution to

$$S_N A_1 - M S_N = NQ_2^\top K_1 \tag{10.23}$$

which in addition satisfies $(D_2 + EK_2)S_N = 0$. Consider now a spectral decomposition $A_1 = Y^{-1} \Lambda Y$, where Λ is the diagonal matrix[3] $\Lambda = \operatorname{diag}(\lambda_1, \ldots \lambda_{n_1})$,

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_{n_1} \end{bmatrix} \quad \text{and} \quad Y^{-1} = \begin{bmatrix} \hat{y}_1 & \ldots & \hat{y}_{n_1} \end{bmatrix}.$$

Then, for fixed N, the unique solution S_N to the Sylvester equation (10.23) can be expressed as (see [7, Thm. 6.5])

$$S_N = \sum_{i=1}^{n_1} (\lambda_i I - M)^{-1} N Q_2^\top K_1 \hat{y}_i y_i$$

[3] Here, $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ denotes the diagonal $n \times n$ matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$.

which implies that $S_N Y^{-1}$ is equal to

$$\left[(\lambda_1 I - M)^{-1} N Q_2^\top K_1 \hat{y}_1 \;\cdots\; (\lambda_{n_1} I - M)^{-1} N Q_2^\top K_1 \hat{y}_{n_1}\right].$$

Note that the matrices $\lambda_i I - M$ are indeed invertible since M is stable and the eigenvalues λ_i of A_1 satisfy $|\lambda_i| \geqslant 1$. Since, in addition, $(D_2 + EK_2)S_N = 0$, we see that for all $i \in [1, n_1]$ we have

$$(D_2 + EK_2)(\lambda_1 I - M)^{-1} N Q_2^\top K_1 \hat{y}_i = 0.$$

Since $Q_2^\top K_1 \neq 0$, there must exist an index i such that $Q_2^\top K_1 \hat{y}_i \neq 0$. For this i, let z be a real vector such that $z^\top Q_2^\top K_1 \hat{y}_i \neq 0$. Now choose $N := e_j z^\top$, where e_j denotes the jth standard basis vector in $\mathbb{R}^{n_2}$. By the discussion above we obtain $(D_2 + EK_2)(\lambda_1 I - M)^{-1} e_j = 0$. Since this holds for any j, we actually find $(D_2 + EK_2)(\lambda_1 I - M)^{-1} = 0$, so $D_2 + EK_2 = 0$. Using (10.22), we must also conclude that $D_1 + EK_1 = 0$, which implies $\operatorname{im} D_1 \subseteq \operatorname{im} E$. Since K_2 is stabilizing it must be of the form $U_- X_{2-}^\sharp$ for some right-inverse $X_{2-}^\sharp$. This implies that $(X_{2+} - A_3 X_{1-})X_{2-}^\sharp$ is stable and $D_2 + EU_- X_{2-}^\sharp = 0$, that is, condition (a) holds. This completes the proof of Theorem 10.8. ☐

Remark 10.9. In order to avoid technicalities, in Theorem 10.8 we have assumed that the matrix A_1 is diagonalizable. The theorem however also holds if we drop this assumption. We omit the proof here.

Remark 10.10. According to Theorem 8, the data are informative for regulator design if and only if at least one of the conditions (a) or (b) holds. Condition (b) is in terms of solvability of the 'data-driven regulator equations' (10.21). These equations hold for all (A_2, B_2) consistent with the data. In the end a matrix S is defined as $S := X_{2-}W$ and together with $V := U_- W$ the classical regulator equations (10.19) are then satisfied for all (A_2, B_2) consistent with the data. This is then 'the classical design', and it can be shown that the difference $x_2(t) - Sx_1(t)$ converges to 0 as t runs off to infinity.

If condition (a) holds, then we can achieve output regulation by making the entire output $z = (D_1 + EK_1)x_1 + (D_2 + EK_2)x_2$ equal to 0 *pointwise in time*. This is done by making $D_1 + EK_1 = 0$ (possible because im $D_1 \subseteq$ im E) and $D_2 + EK_2 = 0$, where $K_2 = U_- X_{2-}^\sharp$ also makes the system endo-stable.

10.6 Illustrative examples

In order to illustrate the theory developed in this chapter up to now, we will give two worked-out examples.

Example 10.11. We will first apply Theorem 10.8 to Example 10.1. Putting the example in our general framework we have

$$x_1 = \begin{bmatrix} r_1 \\ r_2 \\ d \end{bmatrix}, \quad x_2 = x, \quad A_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D_2 = -1, \quad E = 0.$$

Assume $T = 3$, and the data on the disturbance input are

$$D_- = \begin{bmatrix} d(0) & d(1) & d(2) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Since the signal to be tracked is $\cos \frac{1}{2}\pi t$, we must have $r_1(0) = 1$, $r_2(0) = 0$ so $r_1(t) = \cos \frac{1}{2}\pi t$ and $r_2(t) = \cos \frac{1}{2}\pi(t+1)$. This leads to

$$X_{1-} = \begin{bmatrix} r_1(0) & r_1(1) & r_1(2) \\ r_2(0) & r_2(1) & r_2(2) \\ d(0) & d(1) & d(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Assume that $U_- = \begin{bmatrix} u(0) & u(1) & u(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and

$$X_2 = \begin{bmatrix} x_2(0) & x_2(1) & x_2(2) & x_2(3) \end{bmatrix} = \begin{bmatrix} 0 & \frac{3}{2} & 2 & \frac{5}{2} \end{bmatrix}.$$

It can be checked that condition (b) of Theorem 10.8 holds. Indeed, a solution W to the linear equations (10.21) is given by

$$W = \begin{bmatrix} -1 & 1 & -1 \\ \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Furthermore, $X_{2-}^\sharp = \begin{bmatrix} -\frac{1}{2} & \frac{2}{3} & 0 \end{bmatrix}^\top$ is a right-inverse of X_{2-} and

$$(X_{2+} - A_3 X_{1-}) X_{2-}^\sharp = \frac{1}{2}$$

is stable. A regulator is then given by

$$K_1 = U_-(I - X_{2-}^\sharp X_{2-}) W = \begin{bmatrix} -\frac{1}{2} & 1 & -1 \end{bmatrix}$$

and $K_2 := U_- X_{2-}^\sharp = -\frac{1}{2}$.

It can be checked that the above data are consistent with the true endosystem $a_{\text{true}} = 1, b_{\text{true}} = 1$. In fact, in this particular example, the true system is uniquely determined by the data. Indeed, this follows from the fact that

$$X_{2+} = \begin{bmatrix} a_{\text{true}} & b_{\text{true}} \end{bmatrix} \begin{bmatrix} X_{2-} \\ U_- \end{bmatrix} + D_-$$

in which $\begin{bmatrix} X_{2-} \\ U_- \end{bmatrix}$ has full row rank. Thus, a regulator could also have been computed directly from the regulator equations (10.19) after first identfying the true endosystem $a_{\text{true}} = 1, b_{\text{true}} = 1$. It can indeed be verified that $S = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and $V = \begin{bmatrix} -1 & 1 & -1 \end{bmatrix}$ satisfy the regulator equations (10.19) for the true endosystem. By choosing $K_2 = -\frac{1}{2}$, this would then lead to the same regulator as above with $K_1 = -K_2 S + V = \begin{bmatrix} -\frac{1}{2} & 1 & -1 \end{bmatrix}$. ■

We note that, in general, the true endosystem may *not* be uniquely determined by the data. This is illustrated by the following example.

Example 10.12. Consider the two-dimensional endosystem

$$x_2(t+1) = A_{2,\text{true}} x_2(t) + B_{2,\text{true}} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d(t)$$

where $A_{2,\text{true}}$ and $B_{2,\text{true}}$ are unknown 2×2 and 2×1 matrices, respectively. Let $x_2 = \begin{bmatrix} x_{21} & x_{22} \end{bmatrix}^\top$. The disturbance input d is assumed to be a constant signal with finite amplitude, so is generated by $d(t+1) = d(t)$. We want to design a regulator such that $2x_{21} + \frac{1}{2}x_{22}$ tracks a given reference signal. In this example, the reference signals r are assumed to be generated by a given autonomous linear system with state space dimension, say, n_1. Its representation will be irrelevant here. The total exosystem will then have state space dimension $n_1 + 1$, and our output equation is given by $z(t) = D_1 x_1(t) + D_2 x_2(t) + Eu(t)$, with D_1 a $1 \times (n_1 + 1)$ matrix such that $D_1 x_1 = -r$ and $D_2 = \begin{bmatrix} 2 & \frac{1}{2} \end{bmatrix}$. We take $E = 2$. Also note that $A_3 = \begin{bmatrix} 0_{1,n_1} & 0 \\ 0_{1,n_1} & 1 \end{bmatrix}$. Suppose that $T = 2$ and assume we have the following data:

$$U_- = \begin{bmatrix} -1 & -1 \end{bmatrix}, \; D_- = \begin{bmatrix} 1 & 1 \end{bmatrix}, \; X_2 = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 2 & \frac{5}{2} \end{bmatrix}.$$

These data were generated by the true endosystem

$$A_{2,\text{true}} = \begin{bmatrix} 2 & \frac{1}{8} \\ 4 & \frac{5}{4} \end{bmatrix}, \; B_{2,\text{true}} = \begin{bmatrix} \frac{3}{2} \\ 3 \end{bmatrix}.$$

We now check condition (a) of Theorem 10.8. First note that, indeed, $\operatorname{im} D_1 \subseteq \operatorname{im} E$. Also, X_{2-} is non-singular and $(X_{2+} - A_3 X_{1-}) X_{2-}^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ 1 & \frac{1}{2} \end{bmatrix}$. This matrix has eigenvalues $\frac{1}{2} \pm \frac{1}{2}i$, so it is stable. Finally, $D_2 + EU_- X_{2-}^{-1} = 0$. According to Theorem 10.8, a regulator for all endosystems consistent with the given data is given by

$$K_2 = U_1 X_{2-}^{-1} = \begin{bmatrix} -1 & -\frac{1}{4} \end{bmatrix}, \; K_1 = -\frac{1}{2} D_1. \tag{10.24}$$

It can be verified that the set of endosystems consistent with our data is equal to the affine set

$$\Sigma_{\mathcal{D}} = \left\{ \left(\begin{bmatrix} a & \frac{1}{4}a - \frac{3}{8} \\ b & \frac{1}{4}b + \frac{1}{4} \end{bmatrix}, \begin{bmatrix} a - \frac{1}{2} \\ b - 1 \end{bmatrix} \right) \mid a, b \in \mathbb{R} \right\}.$$

The controller given by (10.24) is a regulator for all these endosystems. ■

10.7 Extension: unknown interconnection matrix

In this section we will extend the theory developed in this chapter to the situation that in the endosystem, in addition to A_2 and B_2, also the matrix A_3 that interconnects the endosystem to the exosystem is unknown.

Again, consider the model class of endosystems and exosystems of the form

$$x_2(t+1) = A_2 x(t) + B_2 u(t) + A_3 x_1(t) \tag{10.25}$$
$$x_1(t+1) = A_1 x_1(t). \tag{10.26}$$

We will now consider the situation that all three matrices A_2, B_2 and A_3 are unknown. Again, the matrix A_1 is assumed to be known. The output to be regulated is specified by

$$A_{3,\text{true}} z(t) = D_1 x_1(t) + D_2 x_2(t) + Eu(t) \tag{10.27}$$

where D_1, D_2 and E are assumed to be known as well. Corresponding to the new situation, our new model class $\mathcal{M}$ now consists of all systems given by (10.25), (10.26) and (10.27). The model class $\mathcal{M}$ is parametrized by the matrices A_2, B_2 and A_3. The true (unknown) values of these matrices are given by $A_{2,\text{true}}$, $B_{2,\text{true}}$ and $A_{3,\text{true}}$. As before, we have data given by (10.11), and the true endosystem generates these data, which now means that

$$X_{2+} = A_{2,\text{true}} X_{2-} + B_{2,\text{true}} U_- + A_{3,\text{true}} X_{1-} \,. \tag{10.28}$$

An endosystem with system matrices (A_2, B_2, A_3) is called consistent with these data if also A_2, B_2 and A_3 satisfy the equation

$$X_{2+} = A_2 X_{2-} + B_2 U_- + A_3 X_{1-} \,. \tag{10.29}$$

The set of all (A_2, B_2, A_3) that are consistent with the data is again denoted by $\Sigma_{\mathcal{D}}$, and is now given by

$$\Sigma_{\mathcal{D}} := \{(A_2, B_2, A_3) \mid (10.29) \text{ holds}\} \,. \tag{10.30}$$

As before, we aim at regulator design based on the data (U_-, X_{1-}, X_2), and in order to find a suitable regulator for the true system we should find one that works for all systems (A_2, B_2, A_3) in $\Sigma_{\mathcal{D}}$. This again leads to a concept of informativity, where we note that the model class is now different from the one that we considered before.

Definition 10.13. We say that the data (U_-, X_{1-}, X_2) are *informative for regulator design* if there exists K_1 and K_2 such that the control law $u(t) = K_1 x_1(t) + K_2 x_2(t)$ is a regulator for any endosystem with (A_2, B_2, A_3) in $\Sigma_{\mathcal{D}}$.

As in the scenario with a known interconnection matrix A_3 that was treated in Section 10.5, a necessary condition for the data (U_-, X_{1-}, X_2) to be informative for regulator design is that they are informative for endo-stabilization. For the new model class $\mathcal{M}$ this is now defined as follows.

Definition 10.14. We call the data (U_-, X_{1-}, X_2) *informative for endo-stabilization* if there exists K_2 such that $A_2+B_2K_2$ is a stable matrix for all (A_2, B_2, A_3) in $\Sigma_{\mathcal{D}}$.

The following analogue of Proposition 10.6 will be instrumental in obtaining necessary and sufficient conditions for informativity for endo-stabilization.

Proposition 10.15. *Let T be a positive integer. Let Z, X be real $n \times T$ matrices, let U be a real $m \times T$ matrix and let D be a real $n_1 \times T$ matrix. Consider the set*

$$\Sigma_{(Z,X,U,D)} := \{(A, B, E) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times n_1} \mid Z = AX + BU + ED\}.$$

Assume that $\Sigma_{(Z,X,U,D)}$ is non-empty. Then the following hold:

(a) *There exists a matrix K such that $A + BK$ is stable for all $(A, B, E) \in \Sigma_{(Z,X,U,D)}$ if and only if X has full row rank, and there exists a right-inverse $X^\sharp$ such that $ZX^\sharp$ is stable and $DX^\sharp = 0$. In that case, by taking $K := UX^\sharp$ we have $A + BK$ is stable for all $(A, B, E) \in \Sigma_{(Z,X,U,D)}$.*

(b) *For any K such that $A + BK$ is stable for all $(A, B, E) \in \Sigma_{(Z,X,U,D)}$ there exists a right-inverse $X^\sharp$ such that $K = UX^\sharp$, $DX^\sharp = 0$ and, moreover, $A + BK = ZX^\sharp$ for all $(A, B) \in \Sigma_{(Z,X,U,D)}$.*

Proof. A proof of this proposition is similar to the proof of Lemma 8.4. □

This immediately gives the following conditions for informativity for endo-stabilization.

Lemma 10.16. *The data (U_-, X_{1-}, X_2) are informative for endo-stabilization if and only if X_{2-} has full row rank and it has a right inverse $X_{2-}^\sharp$ such that $X_{2+}X_{2-}^\sharp$ is stable and $X_{1-}X_{2-}^\sharp = 0$. In that case, by taking $K_2 := U_-X_{2-}^\sharp$ we have $A_2 + B_2K_2$ is stable for all $(A_2, B_2, A_3) \in \Sigma_{\mathcal{D}}$.*

The following analogue of Theorem 10.8 then gives necessary and sufficient conditions for informativity for regulator design in the situation that all three matrices representing the endosystem are unknown.

Theorem 10.17. *Assume that A_1 is anti-stable and suppose, for simplicity, that it is diagonalizable. Then the data (U_-, X_{1-}, X_2) are informative for regulator design if and only if at least one of the following two conditions hold:*

(a) *X_{2-} has full row rank and it has a right-inverse $X_{2-}^\sharp$ such that $X_{2+}X_{2-}^\sharp$ is stable, $X_{1-}X_{2-}^\sharp = 0$ and $D_2 + EU_-X_{2-}^\sharp = 0$. Moreover, $\operatorname{im} D_1 \subseteq \operatorname{im} E$. In this case, a regulator is found as follows: choose K_1 such that $D_1+EK_1 = 0$ and define $K_2 := U_-X_{2-}^\sharp$.*

(b) *X_{2-} is right-invertible and it has a right-inverse $X_{2-}^\sharp$ such that $X_{2+}X_{2-}^\sharp$ is stable and $X_{1-}X_{2-}^\sharp = 0$. Moreover, there exists a solution W to the linear equations*

$$X_{2-}WA_1 - X_{2+}W = 0 \tag{10.31a}$$

$$X_{1-}W = I \tag{10.31b}$$

$$D_1 + (D_2X_{2-} + EU_-)W = 0. \tag{10.31c}$$

In this case, a regulator is found as follows: take $K_1 := U_-(I - X_{2-}^\sharp X_{2-})W$ and $K_2 := U_-X_{2-}^\sharp$.

Proof. We first prove sufficiency. Assume that condition (a) holds. Take $K_2 := U_-X_{2-}^\sharp$. Then by Lemma 10.16, $A_2 + B_2K_2$ is stable for all $(A_2, B_2, A_3) \in \Sigma_\mathcal{D}$. Since A_1 is assumed to be anti-stable, this implies that for all $(A_2, B_2, A_3) \in \Sigma_\mathcal{D}$ there exists a unique solution S to the Sylvester equation $SA_1 - (A_2 + B_2K_2)S = A_3 + B_2K_1$. Take K_1 such that $D_1 + EK_1 = 0$. Since also $D_2 + EK_2 = 0$, this solution S also satisfies $D_1 + EK_1 + (D_2 + EK_2)S = 0$. Thus, for all $(A_2, B_2, A_3) \in \Sigma_\mathcal{D}$, there exists a matrix S that satisfies the equations (10.20). It follows from Proposition 10.3 that for all $(A_2, B_2, A_3) \in \Sigma_\mathcal{D}$ the controlled system is endo-stable and output regulated.

Next, assume that condition (b) holds. By Lemma 10.7, the data are informative for endo-stabilization and by taking $K_2 := U_1X_-^\sharp$ we have $A_2 + B_2K_2$ stable for all $(A_2, B_2, A_3) \in \Sigma_\mathcal{D}$. Let W satisfy the equations (10.31). Define $S := X_{2-}W$ and $V := U_-W$. Then the pair (S, V) satisfies the regulator equations (10.19) for all $(A_2, B_2, A_3) \in \Sigma_\mathcal{D}$. By Proposition 10.4, for each such (A_2, B_2, A_3) a regulator is given by the pair (K_1, K_2), with $K_1 = -K_2S + V = -K_2X_{2-}W + U_-W = U_-(I - X_{2-}^\sharp X_{2-})W$. This completes the proof of the sufficiency part.

We will now prove the necessity part. Assume that the data are informative for regulator design. By Proposition 10.3, there exist K_1 and K_2 and for any $(A_2, B_2, A_3) \in \Sigma_{\mathcal{D}}$ a matrix $S_{(A_2,B_2,A_3)}$ such that $A_2 + B_2K_2$ is stable and

$$\begin{aligned} S_{(A_2,B_2,A_3)}A_1 - (A_2 + B_2K_2)S_{(A_2,B_2,A_3)} &= A_3 + B_2K_1 \\ D_1 + EK_1 + (D_2 + EK_2)S_{(A_2,B_2,A_3)} &= 0. \end{aligned}$$

Note that $S_{(A_2,B_2,A_3)}$ may depend on the choice of $(A_2, B_2, A_3) \in \Sigma_{\mathcal{D}}$. However, since $A_2 + B_2K_2$ is stable for all $(A_2, B_2, A_3) \in \Sigma_{\mathcal{D}}$, by Proposition 10.15 there exists a right-inverse $X_{2-}^{\sharp}$ of X_{2-} such that $X_{1-}X_{2-}^{\sharp} = 0$ and $A_2 + B_2K_2 = X_{2+}X_{2-}^{\sharp}$ for all $(A_2, B_2, A_3) \in \Sigma_{\mathcal{D}}$. The latter matrix is independent of (A_2, B_2, A_3). Call it M.

Now define

$$\Sigma_{\mathcal{D}}^0 := \{(A_2^0, B_2^0, A_3^0) \mid \begin{bmatrix} A_2^0 & B_2^0 & A_3^0 \end{bmatrix} \begin{bmatrix} X_{2-} \\ U_- \\ X_{1-} \end{bmatrix} = 0\}.$$

Again, we distinguish two cases, namely (i) $A_3^0 + B_2^0K_1 = 0$ for all (A_2^0, B_2^0, A_3^0) in $\Sigma_{\mathcal{D}}^0$, and (ii) $A_3^0 + B_2^0K_1 \neq 0$ for some $(A_2^0, B_2^0, A_3^0) \in \Sigma_{\mathcal{D}}^0$.

We first consider case (i). Then for all $(A_2, B_2, A_3), (\bar{A}_2, \bar{B}_2, \bar{A}_3) \in \Sigma_{\mathcal{D}}$ we have $A_3 + B_2K_1 = \bar{A}_3 + \bar{B}_2K_1$. Hence there exists a *common* matrix S that solves the equations

$$\begin{aligned} SA_1 - MS &= A_3 + B_2K_1 \\ D_1 + EK_1 + (D_2 + EK_2)S &= 0 \end{aligned}$$

for all $(A_2, B_2, A_3) \in \Sigma_{\mathcal{D}}$. The first of these equation can be written as

$$SA_1 - \begin{bmatrix} A_2 & B_2 & A_3 \end{bmatrix} \begin{bmatrix} S \\ K_2S + K_1 \\ I \end{bmatrix} = 0$$

Since this holds for all $(A_2^0, B_2^0, A_3^0) \in \Sigma_{\mathcal{D}}$, we must have

$$\begin{bmatrix} A_2^0 & B_2^0 & A_3^0 \end{bmatrix} \begin{bmatrix} S \\ K_2S + K_1 \\ I \end{bmatrix} = 0$$

for all $(A_2^0, B_2^0, A_3^0) \in \Sigma_{\mathcal{D}}^0$. This implies

$$\operatorname{im} \begin{bmatrix} S \\ K_2S + K_1 \\ I \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} X_{2-} \\ U_- \\ X_{1-} \end{bmatrix}.$$

Thus, there exists a matrix W such that

$$\begin{bmatrix} S \\ K_2 S + K_1 \\ I \end{bmatrix} = \begin{bmatrix} X_{2-} \\ U_- \\ X_{1-} \end{bmatrix} W.$$

It is then easily seen that W satisfies the equations (10.31), showing that condition (b) holds.

Next, consider case (ii). Let Q be a real $(n_2 + m + n_1) \times r$ matrix such that

$$\ker \begin{bmatrix} X_{2-} \\ U_- \\ X_{1-} \end{bmatrix}^\top = \operatorname{im} Q.$$

Partition $Q = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$. Then $(A_2^0, B_2^0, A_3^0) \in \Sigma_{\mathcal{D}}^0$ if and only if $A_2^0 = NQ_1^\top$, $B_2^0 = NQ_2^\top$ and $A_3^0 = NQ_3^\top$ for some $n_2 \times r$ matrix N. Note that, by hypothesis, $Q_3^\top + Q_2^\top K_1 \neq 0$.

Let $(A_2, B_2, A_3) \in \Sigma_{\mathcal{D}}$. Recall that for any such (A_2, B_2, A_3) there exists a unique $S_{(A_2,B_2,A_3)}$ such that

$$\begin{aligned} S_{(A_2,B_2,A_3)} A_1 - M S_{(A_2,B_2,A_3)} &= A_3 + B_2 K_1 \\ D_1 + EK_1 + (D_2 + EK_2) S_{(A_2,B_2,A_3)} &= 0. \end{aligned} \tag{10.32}$$

Now let N be any real $n_2 \times r$ matrix. Then also $(A_2 + NQ_1^\top, B_2 + NQ_2^\top, A_3 + NQ_3^\top) \in \Sigma_{\mathcal{D}}$. Define

$$S_N := S_{(A_2,B_2,A_3)} - S_{(A_2+NQ_1^\top, B_2+NQ_2^\top, A_3+NQ_3^\top)}.$$

Then clearly S_N is the unique solution to

$$S_N A_1 - M S_N = N(Q_3^\top + Q_2^\top K_1) \tag{10.33}$$

with, in addition, $(D_2 + EK_2) S_N = 0$. Take a spectral decomposition $A_1 = Y^{-1} \Lambda Y$, where Λ is the diagonal matrix $\Lambda = \operatorname{diag}(\lambda_1, \dots \lambda_{n_1})$,

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_{n_1} \end{bmatrix} \quad \text{and} \quad Y^{-1} = \begin{bmatrix} \hat{y}_1 & \dots & \hat{y}_{n_1} \end{bmatrix}.$$

Then, for fixed N, the unique solution S_N to the Sylvester equation (10.33) can be expressed as (see [7, Thm. 6.5])

$$S_N = \sum_{i=1}^{n_1} (\lambda_i I - M)^{-1} N (Q_3^\top + Q_2^\top K_1) \hat{y}_i y_i$$

which implies that $S_N Y^{-1}$ is equal to

$$\left[(\lambda_1 I - M)^{-1}(Q_3^\top + Q_2^\top K_1)\hat{y}_1 \ \cdots \ (\lambda_{n_1} I - M)^{-1}(Q_3^\top + Q_2^\top K_1)\hat{y}_{n_1}\right].$$

Since, in addition, $(D_2 + EK_2)S_N = 0$, we see that for all $i = 1, \ldots, n_1$ we have

$$(D_2 + EK_2)(\lambda_1 I - M)^{-1}(Q_3^\top + NQ_2^\top K_1)\hat{y}_i = 0.$$

Since $Q_3^\top + Q_2^\top K_1 \neq 0$, there must exist an index i such that $(Q_2^\top + Q_2^\top K_1)\hat{y}_i \neq 0$. For this i, let z be a real vector such that $z^\top(Q_3^\top + Q_2^\top K_1)\hat{y}_i \neq 0$. Now choose $N := e_j z^\top$, where e_j denotes the jth standard basis vector in $\mathbb{R}^{n_2}$. By the above we have $(D_2 + EK_2)(\lambda_1 I - M)^{-1} e_j = 0$. Since this holds for any j, we actually find $D_2 + EK_2 = 0$. Using (10.22), we must also conclude that $D_1 + EK_1 = 0$, which implies $\operatorname{im} D_1 \subseteq \operatorname{im} E$. Since K_2 is stabilizing it must be of the form $U_- X_{2-}^\sharp$ for some right-inverse $X_{2-}^\sharp$ with $X_{1-} X_{2-}^\sharp = 0$. This implies that $X_{2+} X_{2-}^\sharp$ is stable and $D_2 + EU_- X_{2-}^\sharp = 0$, that is, condition (a) holds. This completes the proof of Theorem 10.17. □

10.8 Notes and references

In this chapter we have extended the framework of informativity to the classical algebraic regulator problem [42, 54, 55, 79]. The results are based on the paper [161]. For an extensive treatment of the classical regulator problem we refer to [160]. Within this problem, an important role is played by the so-called exosystem that generates reference and disturbance signals. A broad class of relevant reference signals and disturbances (such as step functions, ramps or sinusoids) can be generated by such exosystems in the form of autonomous linear systems.

We note that data-driven regulator design was also studied in [43] and [35], albeit from a rather different perspective. We also mention alternative methods that deal with tracking objectives, such as iterative feedback tuning (IFT) and virtual reference feedback tuning (VRFT) as developed in [72] and [34], respectively. These methods do however not address the classical regulator problem, and are thus quite different from the work in this chapter.

Part III

System Identification and Experiment Design

11

System identification

In this chapter, we will study the problem of system identification from the perspective of data informativity. Recall from Chapter 1 that there are different methods of system identification for linear systems, one of which is *subspace identification*, see Section 1.2.2. As we have seen in that section, subspace identification typically relies on certain persistency of excitation conditions on the input of the system. These conditions are sufficient to be able to identify the system from data, but in general they are not necessary. It turns out that using the data informativity framework one can obtain *necessary and sufficient* conditions on the data under which system identification is possible. These conditions will be discussed in detail in this chapter.

We begin by introducing some terminology concerning input-state-output systems. Consider the input-state-output system

$$x(t+1) = Ax(t) + Bu(t) \tag{11.1a}$$
$$y(t) = Cx(t) + Du(t) \tag{11.1b}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$, with $n \geqslant 0$, $m, p \geqslant 1$. If $n = 0$, we call the system *memoryless*[1]. By convention, memoryless systems are both controllable and observable. For $k \geqslant 0$, we define the *k-th observability matrix* recursively by

$$\Omega_k := \begin{cases} 0_{0,n} & \text{if } k = 0 \\ \begin{bmatrix} \Omega_{k-1} \\ CA^{k-1} \end{bmatrix} & \text{if } k \geqslant 1. \end{cases} \tag{11.2}$$

We define the *lag* $\ell(C, A)$ of the system as the smallest integer $k \geqslant 0$ such that $\operatorname{rank} \Omega_k = \operatorname{rank} \Omega_{k+1}$. Note that $0 \leqslant \ell(C, A) \leqslant n$. Moreover, if $n = 0$, then also $\ell(C, A) = 0$.

[1]Throughout this chapter, we permit matrices to be void. A *void matrix* is a matrix with zero rows and/or zero columns. We denote by $0_{n,0}$ and $0_{0,m}$ respectively the $n \times 0$ and $0 \times m$ void matrices. If M and N are, respectively $p \times q$ and $q \times r$ matrices, MN is a $p \times r$ void matrix if $p = 0$ or $r = 0$ and $MN = 0_{p,r}$ if $p, r \geqslant 1$ and $q = 0$. The rank of a void matrix is defined to be zero.

In what follows, we will identify a system of the form (11.1) with the matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Given $m \geqslant 1$ and $p \geqslant 1$, we define the set of all systems with n state variables as $\mathcal{S}(n) := \mathbb{R}^{(n+p)\times(n+m)}$. We also define the subset of systems with lag ℓ as

$$\mathcal{S}(\ell, n) := \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{S}(n) \mid A \in \mathbb{R}^{n\times n} \text{ and } \ell(C, A) = \ell \right\}. \tag{11.3}$$

Finally, we define the sets

$$\mathcal{S} := \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{S}(n) \mid n \geqslant 0 \right\} \tag{11.4}$$

$$\mathcal{O} := \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{S} \mid (C, A) \text{ is observable} \right\} \tag{11.5}$$

$$\mathcal{M} := \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{O} \mid (A, B) \text{ is controllable} \right\}. \tag{11.6}$$

Two systems $\left[\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right] \in \mathcal{S}(n)$ with $i \in [1, 2]$ are called *isomorphic* if $D_1 = D_2$ and there exists a nonsingular matrix $S \in \mathbb{R}^{n\times n}$ such that

$$A_1 = S^{-1}A_2S, \quad B_1 = S^{-1}B_2, \quad \text{and} \quad C_1 = C_2S.$$

We say that $\mathcal{S}' \subseteq \mathcal{S}(n)$ has the *isomorphism property* if any two systems in $\mathcal{S}'$ are isomorphic. By convention, the empty set has the isomorphism property.

11.1 Problem formulation

Consider the input-state-output system

$$x(t+1) = A_{\text{true}}x(t) + B_{\text{true}}u(t) \tag{11.7a}$$

$$y(t) = C_{\text{true}}x(t) + D_{\text{true}}u(t) \tag{11.7b}$$

where $A_{\text{true}} \in \mathbb{R}^{n_{\text{true}}\times n_{\text{true}}}$, $B_{\text{true}} \in \mathbb{R}^{n_{\text{true}}\times m}$, $C_{\text{true}} \in \mathbb{R}^{p\times n_{\text{true}}}$ and $D_{\text{true}} \in \mathbb{R}^{p\times m}$ are unknown. Also the state-space dimension $n_{\text{true}} \geqslant 0$ is unknown. We refer to (11.7) as the *true system*. We denote its lag by

$$\ell_{\text{true}} := \ell(C_{\text{true}}, A_{\text{true}}).$$

Throughout this chapter, we assume that the true system is minimal, i.e., both controllable and observable.

Let $T \geqslant 1$ and $(u_{[0,T-1]}, y_{[0,T-1]})$ be input-output data generated by (11.7), i.e., there exists a matrix $X_{[0,T]} \in \mathbb{R}^{n_{\text{true}} \times (T+1)}$ such that

$$\begin{bmatrix} X_{[1,T]} \\ Y_{[0,T-1]} \end{bmatrix} = \begin{bmatrix} A_{\text{true}} & B_{\text{true}} \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix} \begin{bmatrix} X_{[0,T-1]} \\ U_{[0,T-1]} \end{bmatrix} . \tag{11.8}$$

In this chapter we examine under which conditions the true system (11.7) can be uniquely determined from the data $(u_{[0,T-1]}, y_{[0,T-1]})$, up to an isomorphism. To formalize this problem below, we first introduce the set of systems consistent with the data.

Consistent systems

A system $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{S}$ is *consistent with the data* $(u_{[0,T-1]}, y_{[0,T-1]})$ if there exist $n \geqslant 0$ and $X_{[0,T]} \in \mathbb{R}^{n \times (T+1)}$ such that

$$\begin{bmatrix} X_{[1,T]} \\ Y_{[0,T-1]} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_{[0,T-1]} \\ U_{[0,T-1]} \end{bmatrix}. \tag{11.9}$$

The set of all systems that are consistent with the data $(u_{[0,T-1]}, y_{[0,T-1]})$ is denoted by $\mathcal{E}$. We will refer to systems in $\mathcal{E}$ as data-consistent systems, or simply consistent systems for short. The subsets of $\mathcal{E}$ consisting of systems with a given state-space dimension (and lag) are denoted by

$$\mathcal{E}(n) := \mathcal{E} \cap \mathcal{S}(n) \quad \text{and} \quad \mathcal{E}(\ell, n) := \mathcal{E} \cap \mathcal{S}(\ell, n).$$

Given a system $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{E}(n)$, we say that $X_{[0,T]} \in \mathbb{R}^{n \times (T+1)}$ is *a state for* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ if (11.9) holds. Moreover, we say that $X_{[0,T]} \in \mathbb{R}^{n \times (T+1)}$ is *a state for the data* $(u_{[0,T-1]}, y_{[0,T-1]})$ if it is a state for some $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{E}(n)$.

It is straightforward to verify that $\mathcal{E}(n)$ and $\mathcal{E}(\ell, n)$ are closed under isomorphisms. In addition, it follows from (11.8) that

$$\begin{bmatrix} A_{\text{true}} & B_{\text{true}} \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix} \in \mathcal{E}(\ell_{\text{true}}, n_{\text{true}}) \subseteq \mathcal{E}(n_{\text{true}}).$$

Informativity for system identification

In what follows, we will work with prior knowledge about the true system (11.7). We capture this mathematically by assuming that

$$\begin{bmatrix} A_{\text{true}} & B_{\text{true}} \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix} \in \mathcal{S}_{\text{pk}},$$

where $\mathcal{S}_{\text{pk}} \subseteq \mathcal{S}$ is a given set. We are now in the position to define the notion of informativity for system identification.

Definition 11.1. The data $(u_{[0,T-1]}, y_{[0,T-1]})$ are *informative for system identification within* $\mathcal{S}_{\text{pk}}$ if

(a) $\mathcal{E} \cap \mathcal{S}_{\text{pk}} = \mathcal{E}(n_{\text{true}}) \cap \mathcal{S}_{\text{pk}}$, and

(b) $\mathcal{E} \cap \mathcal{S}_{\text{pk}}$ has the isomorphism property.

Condition (a) states that all systems consistent with the data and the prior knowledge have n_{true} states, while (b) asserts that any pair of such systems is isomorphic. Therefore, if the data are informative for system identification we can identify n_{true} and the true system up to an isomorphism.

In the rest of the chapter, we assume that *lower* and *upper bounds* on the *true lag* and *true state-space dimension* are given:

$$L_- \leqslant \ell_{\text{true}} \leqslant L_+ \leqslant N_+ \quad \text{and} \quad L_- \leqslant N_- \leqslant n_{\text{true}} \leqslant N_+. \tag{11.10}$$

Therefore, of particular interest are those systems whose lag and state-space dimension are between the given lower and upper bounds:

$$\mathcal{S}_{[L_-,L_+],[N_-,N_+]} := \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{S}(\ell, n) \mid \ell \in [L_-, L_+] \text{ and } n \in [N_-, N_+] \right\},$$

and the set of consistent systems complying with these bounds:

$$\mathcal{E}_{[L_-,L_+],[N_-,N_+]} := \mathcal{E} \cap \mathcal{S}_{[L_-,L_+],[N_-,N_+]}.$$

The main results of this chapter concern informativity of the data for system identification within the prior knowledge class

$$\mathcal{S}_{\text{pk}} = \mathcal{S}_{[L_-,L_+],[N_-,N_+]} \cap \mathcal{M}. \tag{11.11}$$

Connection to the fundamental lemma

Before stating our main results, we first make a connection to Willems' fundamental lemma as discussed in Chapter 1, see Theorem 1.2. The following is a translation of the fundamental lemma to the language we have developed so far. Recall from Definition 1.1 that the input $u_{[0,T-1]}$ is called persistently exciting of order k if the Hankel matrix $H_k(u_{[0,T-1]})$ has full row rank (equal to km).

Proposition 11.2. *Let $(u_{[0,T-1]}, y_{[0,T-1]})$ be generated by the controllable and observable system (11.7), and suppose that $X_{[0,T]} \in \mathbb{R}^{n_{\text{true}} \times (T+1)}$ satisfies (11.8). Suppose that*

$$T \geqslant L_+ + (N_+ + L_+ + 1)m + N_+.$$

If the input $u_{[0,T-1]}$ is persistently exciting of order $N_+ + L_+ + 1$ then the following three statements hold:

(a) *The matrix*

$$\begin{bmatrix} X_{[0,T-L_+-1]} \\ \hdashline H_{L_++1}(u_{[0,T-1]}) \end{bmatrix} = \begin{bmatrix} x(0) & x(1) & \cdots & x(T-L_+-1) \\ \hdashline u(0) & u(1) & \cdots & u(T-L_+-1) \\ \vdots & \vdots & & \vdots \\ u(L_+) & u(L_++1) & \cdots & u(T-1) \end{bmatrix}$$

has full row rank.

(b) *It holds that*

$$\operatorname{rank} \begin{bmatrix} H_{L_++1}(u_{[0,T-1]}) \\ H_{L_++1}(y_{[0,T-1]}) \end{bmatrix} = (L_+ + 1)m + n_{\text{true}}.$$

(c) *The data* $(u_{[0,T-1]}, y_{[0,T-1]})$ *are informative for system identification within* $\mathcal{S}_{[0,L_+],[0,N_+]} \cap \mathcal{M}$.

Note that item (a) of Proposition 11.2 follows from Theorem 1.2 in Chapter 1 by taking $L = L_+ + 1$. Item (b) simply follows from observability of the pair $(C_{\text{true}}\, A_{\text{true}})$ and the fact that $L_+ \geqslant \ell_{\text{true}}$. Finally, the consequence of item (c) is that the data $(u_{[0,T-1]}, y_{[0,T-1]})$ contain sufficient information to identify the true system up to isomorphism, assuming that the input $u_{[0,T-1]}$ is persistently exciting of order $N_+ + L_+ + 1$. We will provide a complete proof of Theorem 1.2 in Section 11.8. This will immediately yield the proofs of Proposition 11.2, items (a) and (b). In addition, a proof of Proposition 11.2 (c) will be provided in Section 11.8.

11.2 Data informativity for system identification

Our ultimate goal is to prove necessary and sufficient conditions for informativity for system identification within $\mathcal{S}_{[L_-,L_+],[N_-,N_+]} \cap \mathcal{M}$. Intuitively, the necessary and sufficient conditions we are after should be a certain rank condition on a Hankel matrix based on the input-output data of a certain depth. The question of which depth is, however, wide open at this stage. It turns out that the right depth is dictated not only by the given upper bounds L_+ and N_+, but also by the *data*. To elaborate further, we first introduce some further notation and terminology.

Data Hankel matrices

For $k \in [1, T]$, we denote the Hankel matrix of depth k constructed from the data $(u_{[0,T-1]}, y_{[0,T-1]})$ by H_k and the matrix obtained from H_k by removal of its last p rows by G_k. In other words,

$$H_k := \begin{bmatrix} H_k(u_{[0,T-1]}) \\ \hdashline H_k(y_{[0,T-1]}) \end{bmatrix} = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-k) \\ \vdots & \vdots & & \vdots \\ u(k-1) & u(k) & \cdots & u(T-1) \\ \hdashline y(0) & y(1) & \cdots & y(T-k) \\ \vdots & \vdots & & \vdots \\ y(k-1) & y(k) & \cdots & y(T-1) \end{bmatrix}$$

$$G_k := \begin{bmatrix} H_k(u_{[0,T-1]}) \\ \hdashline H_{k-1}(y_{[0,T-2]}) \end{bmatrix} = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-k) \\ \vdots & \vdots & & \vdots \\ u(k-1) & u(k) & \cdots & u(T-1) \\ \hdashline y(0) & y(1) & \cdots & y(T-k) \\ \vdots & \vdots & & \vdots \\ y(k-2) & y(k-1) & \cdots & y(T-2) \end{bmatrix}. \tag{11.12}$$

Here, we note that by convention, $H_0(y_{[0,T-2]}) = 0_{0,T-k+1}$ so that G_1 is simply equal to $H_1(u_{[0,T-1]})$. Furthermore, note that

$$H_k \in \mathbb{R}^{k(m+p)\times(T-k+1)} \quad \text{and} \quad G_k \in \mathbb{R}^{(k(m+p)-p)\times(T-k+1)}.$$

Next, we define for $k \in [0, T]$ the integers

$$\delta_k := \begin{cases} p & \text{if } k = 0 \\ \operatorname{rank} H_k - \operatorname{rank} G_k & \text{if } k \in [1, T]. \end{cases} \tag{11.13}$$

Note that

$$p \geqslant \delta_k \geqslant 0 \text{ for all } k \in [0, T]. \tag{11.14}$$

In the rest of this chapter, we will assume that the input data samples are not all equal to zero, i.e.,

$$U_{[0,T-1]} \neq 0_{m,T}.$$

From this assumption, it follows that $\operatorname{rank} H_T = \operatorname{rank} G_T = 1$ and hence

$$\delta_T = 0. \tag{11.15}$$

The integers δ_k will play a significant role in the later development. In what follows, we employ these integers to present several intermediate results, which will eventually provide necessary and sufficient conditions for data informativity for system identification.

Lags and state-space dimensions of data-consistent systems

Now, we present our first intermediate result establishing bounds on the lag and state-space dimension of *any* data-consistent system in terms of the integers δ_k. To proceed, we let $q \in [0, T-1]$ be the smallest integer such that $\delta_{q+1} = 0$, that is

$$q := \min\{k \in [0, T-1] \mid \delta_{k+1} = 0\}. \tag{11.16}$$

Note that q is well-defined due to (11.15). In the next theorem, we relate q to the lag ℓ of any system consistent with the data. In this theorem and throughout the rest of the chapter, we will use the following convention. For the sum $\sum_{i=1}^{k} a_i$ of real numbers $a_i \in \mathbb{R}$, we say $\sum_{i=1}^{k} a_i = 0$ whenever $k = 0$.

Theorem 11.3. *Suppose that $\mathcal{E}(\ell, n) \neq \varnothing$. Then, the following statements hold:*

(a) *If $T \geqslant \ell + 1$, then $\ell \geqslant q$.*

(b) *If $\ell \geqslant q$, then $n - \sum_{i=1}^{q} \delta_i \geqslant \ell - q$.*

The proof of this theorem is given in Section 11.4.3.

Constructing a consistent system

Our second intermediate result concerns the question how to construct *a* system consistent with the data from the given data $(u_{[0,T-1]}, y_{[0,T-1]})$.

By (11.9), it follows that $X_{[0,T]}$ is a state for the data $(u_{[0,T-1]}, y_{[0,T-1]})$ if and only if

$$\operatorname{rsp}\begin{bmatrix} X_{[1,T]} \\ Y_{[0,T-1]} \end{bmatrix} \subseteq \operatorname{rsp}\begin{bmatrix} X_{[0,T-1]} \\ U_{[0,T-1]} \end{bmatrix}, \tag{11.17}$$

where we recall that the row space of a matrix M is denoted by $\operatorname{rsp} M$. From these definitions, it is clear that a data-consistent system can be obtained by solving the linear equations (11.9) once a state for the data is constructed. In the following, we show that for any data set $(u_{[0,T-1]}, y_{[0,T-1]})$ one can *always* construct a specific consistent system with $\sum_{i=1}^{q} \delta_i$ state variables. Moreover, all consistent systems with $\sum_{i=1}^{q} \delta_i$ state variables have the same lag q.

Theorem 11.4. $\varnothing \neq \mathcal{E}(\sum_{i=1}^{q} \delta_i) = \mathcal{E}(q, \sum_{i=1}^{q} \delta_i)$.

The constructive proof of this theorem (given in Section 11.5.2) provides a novel iterative scheme to create a state from given input-output data.

The shortest lag and the minimum number of states

We define the *shortest lag* $\ell_{\min}$ and the *minimum number of states* $n_{\min}$ of any system consistent with the data $(u_{[0,T-1]}, y_{[0,T-1]})$ as follows:

$$\ell_{\min} := \min\{\ell \geqslant 0 \mid \exists\, n \geqslant 0 \text{ such that } \mathcal{E}(\ell, n) \neq \varnothing\} \tag{11.18}$$

$$n_{\min} := \min\{n \geqslant 0 \mid \mathcal{E}(n) \neq \varnothing\}. \tag{11.19}$$

These integers admit a remarkably simple characterization in terms of δ_k and q, as defined earlier.

Theorem 11.5. *It holds that $\ell_{\min} = q$ and $n_{\min} = \sum_{i=1}^{\ell_{\min}} \delta_i$. Further, $\mathcal{E}(n_{\min}) = \mathcal{E}(\ell_{\min}, n_{\min}) \subset \mathcal{O}$.*

Proof. From Theorem 11.4, we see that $q \geqslant \ell_{\min}$ and $\sum_{i=1}^{q} \delta_i \geqslant n_{\min}$. To prove the first part of the theorem, it is enough to show the reverse inequalities hold. To do so, note that $T \geqslant q+1$ due to the definition of q in (11.16). As such, we see that $T \geqslant \ell_{\min} + 1$. Since $\mathcal{E}(\ell_{\min}, n) \neq \varnothing$ for some n due to the definition of $\ell_{\min}$ in (11.18), Theorem 11.3.(a) yields $\ell_{\min} \geqslant q$ and hence $\ell_{\min} = q$. Due to the definition of $n_{\min}$ in (11.19), $\mathcal{E}(\ell, n_{\min}) \neq \varnothing$ for some ℓ. Then, Theorem 11.3.(b) implies that $n_{\min} - \sum_{i=1}^{\ell_{\min}} \delta_i \geqslant \ell - \ell_{\min} \geqslant 0$. This proves $n_{\min} \geqslant \sum_{i=1}^{\ell_{\min}} \delta_i$ and hence $n_{\min} = \sum_{i=1}^{\ell_{\min}} \delta_i$.

To prove the second claim, first observe that $\mathcal{E}(n_{\min}) = \mathcal{E}(\ell_{\min}, n_{\min})$ readily follows from Theorem 11.4. Therefore, it remains to prove that $\mathcal{E}(n_{\min}) \subset \mathcal{O}$. To do so, suppose, on the contrary, that there is an unobservable system in $\mathcal{E}(n_{\min})$. Then, a straightforward Kalman decomposition argument yields a consistent system with state-space dimension strictly less than $n_{\min}$. This would, however, contradict the definition of $n_{\min}$ in (11.19). Consequently, all systems in $\mathcal{E}(n_{\min})$ are observable, that is $\mathcal{E}(n_{\min}) \subset \mathcal{O}$. □

Sharpening the upper bound on the true lag

Let ℓ and n be the lag and state-space dimension of some consistent system. Then, combining Theorems 11.3.(b) and 11.5 leads to the following immediate but rather crucial inequality:

$$\ell \leqslant n - n_{\min} + \ell_{\min}. \tag{11.20}$$

In particular, this inequality implies that

$$L_+^{\mathrm{d}} := N_+ - n_{\min} + \ell_{\min}$$

is an upper bound for the lag of *every* consistent system with at most N_+ states. This upper bound, which is purely determined by the data and N_+, may help us to sharpen the upper bound on the lag, L_+. Indeed, we can replace L_+ by the *actual* upper bound on the lag

$$L_+^{\mathrm{a}} := \min(L_+, L_+^{\mathrm{d}})$$

since it follows from (11.20) that

$$\mathcal{E}_{[L_-,L_+],[N_-,N_+]} = \mathcal{E}_{[L_-,L_+^{\mathrm{a}}],[N_-,N_+]}. \tag{11.21}$$

Necessary and sufficient conditions

The following theorem provides necessary and sufficient conditions for the data to be informative for system identification.

Theorem 11.6. *The data $(u_{[0,T-1]}, y_{[0,T-1]})$ are informative for system identification within $\mathcal{S}_{[L_-,L_+],[N_-,N_+]} \cap \mathcal{M}$ if and only if the following conditions hold:*

$$\ell_{\min} \geqslant L_- \tag{11.22a}$$
$$n_{\min} \geqslant N_- \tag{11.22b}$$
$$T \geqslant L_+^{\mathrm{a}} + (L_+^{\mathrm{a}} + 1)m + n_{\min} \tag{11.22c}$$
$$\operatorname{rank} H_{L_+^{\mathrm{a}}+1} = (L_+^{\mathrm{a}} + 1)m + n_{\min}. \tag{11.22d}$$

Moreover, if the conditions in (11.22) are satisfied, then

$$\ell_{\mathrm{true}} = \ell_{\min} \tag{11.23a}$$
$$n_{\mathrm{true}} = n_{\min} \tag{11.23b}$$
$$\mathcal{E}_{[L_-,L_+],[N_-,N_+]} \cap \mathcal{M} = \mathcal{E}(n_{\min}). \tag{11.23c}$$

This theorem presents a *truly data-based* necessary and sufficient condition for informativity for system identification. Indeed, one only needs to compute $\ell_{\min}$ and $n_{\min}$ directly from the data via Theorem 11.5 to verify the presented conditions.

Several remarks discussing the consequences of Theorem 11.6 and its relation to existing results are in order.

Remark 11.7. As one might expect, informativity for system identification requires a certain rank condition on data Hankel matrices. What is truly remarkable, however, is that the depth of the Hankel matrix, which plays a pivotal role in determining whether the data are rich enough for system identification, depends *not only* on the prior knowledge of the system *but also* on the given data. Indeed, we recall that L_+^{a} depends on $n_{\min}$ and $\ell_{\min}$, which are, in turn, computed using the data.

Remark 11.8. Theorem 11.6 is applicable even if one of the upper bounds is not explicitly known. Indeed, if only the upper bound L_+ on the lag is known but an upper bound on the state-space dimension is not, one can fix $N_+ = pL_+$ since $n \leqslant p\ell$ for any observable system in $\mathcal{S}(\ell, n)$. For the choice $N_+ = pL_+$, note that

$$N_+ - n_{\min} + \ell_{\min} \geqslant pL_+ - p\ell_{\min} + \ell_{\min} \geqslant L_+$$

where the first inequality follows from the fact that $n_{\min} \leqslant p\ell_{\min}$ as $\mathcal{E}(\ell_{\min}, n_{\min}) \subset \mathcal{O}$ due to Theorem 11.5, while the second inequality follows from $L_+ \geqslant \ell_{\text{true}} \geqslant \ell_{\min}$. As such, if $N_+ = pL_+$ then $L_+^{\text{a}} = L_+$.

Conversely, if the upper bound N_+ on the state-space dimension is known but an upper bound on the lag is not, one can fix $L_+ = N_+$ since the lag of a system cannot exceed its state-space dimension. For the choice $N_+ = L_+$, note that

$$L_+ \geqslant L_+ - n_{\min} + \ell_{\min} = N_+ - n_{\min} + \ell_{\min}$$

where the inequality follows from the fact that $n_{\min} \geqslant \ell_{\min}$. Therefore, if $L_+ = N_+$ then $L_+^{\text{a}} = L_+^{\text{d}}$.

Remark 11.9. Proposition 11.2 can be applied only if the data length is *at least* $L_+ + (N_+ + L_+ + 1)m + N_+$ whereas Theorem 11.6 can be applied if the data length is *at least* $L_+^{\text{a}} + (L_+^{\text{a}} + 1)m + n_{\text{true}}$. The difference between these data lengths can be significant, as illustrated further in Section 11.3.

Remark 11.10. Even though it is not explicit in the statement of Theorem 11.6, one can utilize (11.23c) and the constructive proof of Theorem 11.4 to build, from given informative data, a consistent system isomorphic to the true system. This will be illustrated by an example after the proof of Theorem 11.4 in Section 11.5.3.

Remark 11.11. The lower bounds of the true lag and state-space dimension do not play any role in informativity of the data. To see this, note that if the data are informative within $\mathcal{S}_{[L_-,L_+],[N_-,N_+]} \cap \mathcal{M}$ then we have necessarily $\ell_{\text{true}} = \ell_{\min}$ and $n_{\text{true}} = n_{\min}$ due to (11.23a) and (11.23b). As such, the same data are informative within $\mathcal{S}_{[0,L_+],[0,N_+]} \cap \mathcal{M}$. Conversely, if the data are informative within $\mathcal{S}_{[0,L_+],[0,N_+]} \cap \mathcal{M}$ so they are within $\mathcal{S}_{[L_-,L_+],[N_-,N_+]} \cap \mathcal{M}$ since the latter set is a subset of the former.

Before proceeding to prove the main results, we illustrate them by an example.

11.3 Illustrative example

Consider the true system (11.7) where $n_{\text{true}} = 3$, $m = 2$, $p = 2$, and

$$\begin{bmatrix} A_{\text{true}} & B_{\text{true}} \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix} = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right].$$

Note that $\ell_{\text{true}} = 2$. Consider the input-output data

$$\begin{bmatrix} U_{[0,13]} \\ \hline Y_{[0,13]} \end{bmatrix} = \left[\begin{array}{cccccccccccccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline 2 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 3 & 2 & 2 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \end{array}\right].$$

One can verify that (11.8) is satisfied with the state data

$$X_{[0,14]} = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 1 & 2 & 2 & 3 & 2 & 2 & 2 & 2 & 3 & 2 \\ 1 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

As such, the input-output data are generated by the true system. Table 11.1 presents the values of δ_k integers, ℓ_{min}, and n_{min} for different choices of T. The values of δ_k that are not indicated in the table are zero, i.e., $\delta_k = 0$ for every $T \in [4, 14]$ and $k \in [4, T]$.

T	1	2	$[3,5]$	$[6,14]$
δ_0	2	2	2	2
δ_1	0	1	2	2
δ_2		0	0	1
δ_3			0	0
ℓ_{min}	0	1	1	2
n_{min}	0	1	2	3

Table 11.1: δ_k integers, ℓ_{min}, and n_{min} for different choices of T

In case $L_+ = \ell_{\text{true}}$ and $N_+ = n_{\text{true}}$, a necessary condition for informativity is that $T \geqslant \ell_{\text{true}} + (\ell_{\text{true}}+1)m + n_{\text{true}}$ due to the condition (11.22c) of Theorem 11.6. Therefore, we see that the data $(u_{[0,T-1]}, y_{[0,T-1]})$ cannot be informative if $T <$

11 for any choice of L_+ and N_+. Table 11.2 indicates the informativity of the data (by the symbol '✓') inferred by applying Theorem 11.6 for different choices of L_+, N_+, and T. As mentioned in Remark 11.9, the fundamental lemma requires significantly more data than Theorem 11.6. Indeed, the input $u_{[0,T-1]}$ is not persistently exciting of sufficiently high order for any of the values of T reported in Table 11.2. For example, in the case that $L_+ = 2$ and $N_+ = 3$, the excitation condition of the fundamental lemma requires $T \geqslant L_+ + (N_+ + L_+ + 1)m + N_+ = 17$ samples. In the case that $L_+ = N_+ = 4$, inferring informativity through the fundamental lemma requires at least 26 data points. Note that in the former case, informativity can already be verified via Theorem 11.6 using $T = 11$ samples, while in the latter case this is possible using $T = 14$ samples.

				T			
L_+	N_+	L_+^{d}	L_+^{a}	11	12	13	14
2	3	2	2	✓	✓	✓	✓
2	4	3	2	✓	✓	✓	✓
2	5	4	2	✓	✓	✓	✓
2	6	5	2	✓	✓	✓	✓
3	3	2	2	✓	✓	✓	✓
3	4	3	3				✓
3	5	4	3				✓
3	6	5	3				✓
4	4	3	3				✓

Table 11.2: Informativity of the data for different bounds and T

11.4 Lag structures of consistent systems

In this section we provide a proof of Theorem 11.3. To do this, we need to introduce the notion of lag structure and relate it to the integers δ_k.

11.4.1 The lag structure of a system

Let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{S}(\ell, n)$. Recall that the depth-$k$ observability matrix Ω_k was defined in (11.2). Now, for $k \geqslant 0$, define the integers ρ_k by

$$\rho_k := \begin{cases} p & \text{if } k = 0 \\ \operatorname{rank} \Omega_k - \operatorname{rank} \Omega_{k-1} & \text{if } k \geqslant 1. \end{cases}$$

We refer to the sequence $(\rho_k)_{k\in\mathbb{N}}$ as the *lag structure* of the system $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Note that the integers ρ_k for $k \in \mathbb{N}$ are related to a *specific* system. If necessary to resolve ambiguities, we use the notation $\rho_k(C, A)$.

The following properties of lag structures will be employed later.

Lemma 11.12. *Let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{S}(\ell, n)$ and let $(\rho_k)_{k\in\mathbb{N}}$ be its lag structure. The following statements hold:*

(a) $p \geqslant \rho_k \geqslant 0$ *for all* $k \geqslant 0$.

(b) $\rho_\ell \geqslant 1$ *and* $\rho_{k+1} = 0$ *for all* $k \geqslant \ell$.

(c) $n \geqslant \sum_{i=1}^{\ell} \rho_i$ *and the inequality holds as equality if* (C, A) *is observable.*

(d) $\rho_k \geqslant \rho_{k+1}$ *for all* $k \geqslant 0$.

Proof. The statements (a) and (b) readily follow from the definitions of ρ_k and the lag. To prove (c), note that $\operatorname{rank}\Omega_\ell = \sum_{i=1}^{\ell} \rho_i$. This proves (c) since $n \geqslant \operatorname{rank}\Omega_\ell$ and the inequality holds as equality if the pair (C, A) is observable.

To prove (d), we first note that the statement follows from (a) if $k = 0$. Therefore, in what follows let $k \geqslant 1$. Note that $\operatorname{rsp}\Omega_k = \operatorname{rsp}\Omega_{k-1} + \operatorname{rsp} CA^{k-1}$ since

$$\Omega_k = \begin{bmatrix} \Omega_{k-1} \\ CA^{k-1} \end{bmatrix}.$$

As such, we have $\operatorname{rank}\Omega_k = \operatorname{rank}\Omega_{k-1} + \operatorname{rank} CA^{k-1} - \dim \mathcal{V}_k$ where $\mathcal{V}_k = \operatorname{rsp}\Omega_{k-1} \cap \operatorname{rsp} CA^{k-1}$. This leads to the following alternative characterization of ρ_k:

$$\rho_k = \operatorname{rank} CA^{k-1} - \dim \mathcal{V}_k. \tag{11.24}$$

Now, observe that $CA^k = CA^{k-1}A$. Application of the rank-nullity theorem to the latter yields

$$\operatorname{rank} CA^{k-1} = \operatorname{rank} CA^k + \dim \mathcal{W}_k \tag{11.25}$$

where $\mathcal{W}_k = \operatorname{rsp} CA^{k-1} \cap \operatorname{lker} A$, and where we recall that $\operatorname{lker} M = \{x \in \mathbb{R}^{1\times m} \mid xM = 0\}$ denotes the left kernel of a matrix $M \in \mathbb{R}^{m\times n}$. By combining (11.24) and (11.25), we obtain

$$\rho_k - \rho_{k+1} = \dim \mathcal{W}_k + \dim \mathcal{V}_{k+1} - \dim \mathcal{V}_k.$$

Let $\mathcal{Z}_k$ be a subspace such that $\mathcal{V}_k = (\mathcal{V}_k \cap \mathcal{W}_k) \oplus \mathcal{Z}_k$. Then, we have

$$\rho_k - \rho_{k+1} \geqslant \dim \mathcal{V}_{k+1} - \dim \mathcal{Z}_k. \tag{11.26}$$

Let $d = \dim \mathcal{Z}_k$ and η_i with $i \in [1, d]$ be a basis for $\mathcal{Z}_k$. From the definition of $\mathcal{Z}_k$, it readily follows that $\eta_i A$ are linearly independent and

$$\eta_i A \in \operatorname{rsp}\Omega_{k-1}A \cap \operatorname{rsp} CA^k \subseteq \operatorname{rsp}\Omega_k \cap \operatorname{rsp} CA^k = \mathcal{V}_{k+1}.$$

Therefore, $\dim \mathcal{Z}_k \leqslant \dim \mathcal{V}_{k+1}$. Hence, it follows from (11.26) that $\rho_k \geqslant \rho_{k+1}$. This proves the lemma. □

11.4.2 Lag structures and δ_k integers

Let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{S}(n)$. For $k \geqslant 0$, define the *k-th controllability matrix*, and the *k-th system matrix*, respectively, by

$$\Gamma_k := \begin{cases} 0_{n,0} & \text{if } k = 0 \\ \begin{bmatrix} A^{k-1}B & \Gamma_{k-1} \end{bmatrix} & \text{if } k \geqslant 1 \end{cases} \tag{11.27}$$

$$\Theta_k := \begin{cases} 0_{0,0} & \text{if } k = 0 \\ \begin{bmatrix} \Theta_{k-1} & 0 \\ C\Gamma_{k-1} & D \end{bmatrix} & \text{if } k \geqslant 1. \end{cases} \tag{11.28}$$

Note that the system matrix Θ_k is a block Toeplitz matrix constructed from the first k Markov parameters of the system (11.1).

Given a state sequence $X_{[0,T]}$ for a consistent system $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{E}(n)$, the data Hankel matrices H_k and G_k are related to the observability and system matrices as follows:

$$H_k = \begin{bmatrix} H_k(u_{[0,T-1]}) \\ H_k(y_{[0,T-1]}) \end{bmatrix} = \Phi_k \begin{bmatrix} X_{[0,T-k]} \\ H_k(u_{[0,T-1]}) \end{bmatrix} \tag{11.29a}$$

$$G_k = \begin{bmatrix} H_k(u_{[0,T-1]}) \\ H_{k-1}(Y_{[0,T-2]}) \end{bmatrix} = \Psi_k \begin{bmatrix} X_{[0,T-k]} \\ H_k(u_{[0,T-1]}) \end{bmatrix} \tag{11.29b}$$

where

$$\Phi_k := \begin{bmatrix} 0 & I \\ \Omega_k & \Theta_k \end{bmatrix} \text{ and } \Psi_k := \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I_m \\ \Omega_{k-1} & \Theta_{k-1} & 0 \end{bmatrix}. \tag{11.30}$$

The following result relates the integers δ_k (defined in terms of the data matrices H_k and G_k only) and the integers ρ_k (defined by the matrices C and A of a specific consistent system).

Lemma 11.13. *Let $(\rho_k)_{k\in\mathbb{N}}$ be the lag structure of a consistent system. For every $k \in [1, T]$, $\rho_k \geqslant \delta_k$.*

Proof. Let $k \in [1, T]$. By applying the rank-nullity theorem to (11.29), we see that

$$\operatorname{rank} H_k = \operatorname{rank} \Phi_k - \dim(\operatorname{rsp} \Phi_k \cap \operatorname{lker} J_k) \tag{11.31}$$

$$\operatorname{rank} G_k = \operatorname{rank} \Psi_k - \dim(\operatorname{rsp} \Psi_k \cap \operatorname{lker} J_k) \tag{11.32}$$

where

$$J_k := \begin{bmatrix} X_{[0,T-k]} \\ H_k(u_{[0,T-1]}) \end{bmatrix}.$$

Since $\operatorname{rank}\Phi_k - \operatorname{rank}\Psi_k = \operatorname{rank}\Omega_k - \operatorname{rank}\Omega_{k-1}$, subtracting (11.32) from (11.31) yields that $\operatorname{rank} H_k - \operatorname{rank} G_k$ is equal to

$$\operatorname{rank}\Omega_k - \Omega_{k-1} - \dim(\operatorname{rsp}\Phi_k \cap \operatorname{lker} J_k) + \dim(\operatorname{rsp}\Psi_k \cap \operatorname{lker} J_k). \tag{11.33}$$

Due to the definitions in (11.30), $\operatorname{rsp}\Psi_k \subseteq \operatorname{rsp}\Phi_k$ and therefore we have that $\dim(\operatorname{rsp}\Psi_k \cap \operatorname{lker} J_k) \leqslant \dim(\operatorname{rsp}\Phi_k \cap \operatorname{lker} J_k)$. We conclude from the expression for $\operatorname{rank} H_k - \operatorname{rank} G_k$ in (11.33) that $\rho_k \geqslant \delta_k$. This proves the lemma. □

Now, we are ready to prove Theorem 11.3.

11.4.3 Proof of Theorem 11.3

Let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{E}(\ell, n)$ and let $(\rho_k)_{k\in\mathbb{N}}$ be the lag structure of the system $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. To prove (a), note that $\rho_{\ell+1} = 0$ due to Lemma 11.12.(b). Since $T-1 \geqslant \ell$ by hypothesis, Lemma 11.13 and (11.14) imply that $\delta_{\ell+1} = 0$. Then, we see that $\ell \geqslant q$ from the definition of q in (11.16).

To prove (b), note that

$$n - \sum_{i=1}^{q} \delta_i \geqslant \sum_{i=1}^{\ell} \rho_i - \sum_{i=1}^{q} \delta_i \tag{11.34}$$

due to Lemma 11.12.(c). Therefore, (b) readily follows from Lemma 11.13 in case $\ell = q$. Suppose that $\ell > q$. Note that

$$\sum_{i=1}^{\ell} \rho_i - \sum_{i=1}^{q} \delta_i \geqslant \sum_{i=q+1}^{\ell} \rho_i \geqslant \ell - q$$

where the first inequality follows from Lemma 11.13, and the second from the statements (b) and (d) of Lemma 11.12. Consequently, (b) follows from (11.34). ■

11.5 State construction

In this section, we present an iterative method to construct a state sequence from given input-output data. This approach distinguishes itself from state reconstruction methods in subspace identification [115] by being applicable to *any* data set (without requiring rank conditions). Apart from being interesting by itself, the procedure will be instrumental in proving Theorem 11.4.

11.5.1 On the left kernels of data Hankel matrices

The integers δ_k obtained from data via (11.13) are intimately related to certain subspaces of the left kernels of data Hankel matrices. To elaborate further, we first introduce a *shift* operator on subspaces. Let $\mathcal{V} \subseteq \mathbb{R}^{1\times\kappa(m+p)}$ be a subspace where $\kappa \in \mathbb{Z}_+$. Define $\sigma\mathcal{V}$ as the subspace of all vectors of the form

$$\begin{bmatrix} 0_{1\times m} & v_1 & 0_{1\times p} & v_2 \end{bmatrix}$$

where $v_1 \in \mathbb{R}^{1\times\kappa m}$ and $v_2 \in \mathbb{R}^{1\times\kappa p}$ satisfy

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \in \mathcal{V}.$$

By convention, $\sigma^0\mathcal{V} := \mathcal{V}$ and $\sigma^k\mathcal{V} = \sigma(\sigma^{k-1}\mathcal{V})$ for $k \geqslant 1$. In what follows, we will use the shorthand notation $\mathbf{0}_p := \{0_{1\times p}\}$. The definitions of H_k and G_k readily yield

$$\operatorname{lker} G_k \times \mathbf{0}_p \subseteq \operatorname{lker} H_k \tag{11.35}$$

for all $k \in [1, T]$. Further, it follows from the definition of σ and the Hankel structure that if $T \geqslant 2$, then for all $k \in [1, T-1]$

$$\sigma \operatorname{lker} H_k \subseteq \operatorname{lker} H_{k+1} \tag{11.36}$$

$$\sigma(\operatorname{lker} G_k \times \mathbf{0}_p) \subseteq \operatorname{lker} G_{k+1} \times \mathbf{0}_p \tag{11.37}$$

$$\sigma \operatorname{lker} H_k \cap (\operatorname{lker} G_{k+1} \times \mathbf{0}_p) = \sigma(\operatorname{lker} G_k \times \mathbf{0}_p). \tag{11.38}$$

The following result shows that $\operatorname{lker} H_k$ can be written into a direct sum of $\operatorname{lker} G_k \times \mathbf{0}_p$ and shifts of certain subspaces.

Lemma 11.14. *For $k \in [1, T]$, there exist subspaces $\mathcal{S}_k \subseteq \mathbb{R}^{1\times k(m+p)}$ satisfying*

$$\operatorname{lker} H_k = \left(\bigoplus_{i=1}^{k} \sigma^{k-i}\mathcal{S}_i \right) \oplus (\operatorname{lker} G_k \times \mathbf{0}_p) \tag{11.39}$$

and $\dim \mathcal{S}_k = \delta_{k-1} - \delta_k$.

Proof. First, we want to prove the existence of subspaces satisfying (11.39). This will be done by induction on k. For $k = 1$, we see from (11.35) that there exists a subspace $\mathcal{S}_1 \subseteq \mathbb{R}^{1\times(m+p)}$ such that

$$\operatorname{lker} H_1 = \mathcal{S}_1 \oplus (\operatorname{lker} G_1 \times \mathbf{0}_p).$$

If $T = 1$, there is nothing more to prove. Suppose that $T \geqslant 2$. Let $k \in [1, T-1]$ and assume that there exist subspaces $\mathcal{S}_i \subseteq \mathbb{R}^{1\times i(m+p)}$ with $i \in [1, k]$ satisfying

$$\operatorname{lker} H_k = \left(\bigoplus_{i=1}^{k} \sigma^{k-i}\mathcal{S}_i \right) \oplus (\operatorname{lker} G_k \times \mathbf{0}_p). \tag{11.40}$$

It follows from (11.35) and (11.36) that

$$\sigma\, \text{lker}\, H_k + (\text{lker}\, G_{k+1} \times \mathbf{0}_p) \subseteq \text{lker}\, H_{k+1}.$$

Therefore, there exists a subspace $\mathcal{S}_{k+1} \subseteq \mathbb{R}^{1\times(k+1)(m+p)}$ such that

$$\text{lker}\, H_{k+1} = \mathcal{S}_{k+1} \oplus (\sigma\, \text{lker}\, H_k + (\text{lker}\, G_{k+1} \times \mathbf{0}_p)) \,. \tag{11.41}$$

Given subspaces $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ with $\mathcal{V}_1 \cap \mathcal{V}_2 = \{0\}$ and $(\mathcal{V}_1 + \mathcal{V}_2) \cap \mathcal{V}_3 = \mathcal{V}_2$, it holds that $\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 = \mathcal{V}_1 \oplus \mathcal{V}_3$. Take

$$\mathcal{V}_1 = \sigma(\bigoplus_{i=1}^{k} \sigma^{k-i}\mathcal{S}_i), \quad \mathcal{V}_2 = \sigma(\text{lker}\, G_k \times \mathbf{0}_p), \quad \text{and} \quad \mathcal{V}_3 = \text{lker}\, G_{k+1} \times \mathbf{0}_p.$$

Note that $\mathcal{V}_1 \cap \mathcal{V}_2 = \{0\}$ due to (11.40) and $(\mathcal{V}_1 + \mathcal{V}_2) \cap \mathcal{V}_3 = \mathcal{V}_2$ due to (11.38). Therefore, we see from (11.40) and (11.41) that

$$\text{lker}\, H_{k+1} = \left(\bigoplus_{i=1}^{k+1} \sigma^{k+1-i}\mathcal{S}_i \right) \oplus (\text{lker}\, G_{k+1} \times \mathbf{0}_p) \,.$$

This proves, by induction on k, that there exist subspaces $\mathcal{S}_k$ such that (11.39) holds. To complete the proof, it remains to show that $\dim \mathcal{S}_k = \delta_{k-1} - \delta_k$ for $k \in [1, T]$. Let $k \in [1, T]$ and observe that (11.39) yields

$$\dim \text{lker}\, H_k = \sum_{i=1}^{k} \dim \mathcal{S}_i + \dim \text{lker}\, G_k.$$

From the rank-nullity theorem, we conclude that

$$\sum_{i=1}^{k} \dim \mathcal{S}_i = p - \delta_k = \delta_0 - \delta_k.$$

Therefore, $\dim \mathcal{S}_1 = \delta_0 - \delta_1$ and

$$\dim \mathcal{S}_k = \sum_{i=1}^{k} \dim \mathcal{S}_i - \sum_{i=1}^{k-1} \dim \mathcal{S}_i = \delta_{k-1} - \delta_k$$

for all $k \in [2, T]$. □

Now, we are ready to prove Theorem 11.4.

11.5.2 Proof of Theorem 11.4

We first show that $\mathcal{E}(\sum_{i=1}^{q}\delta_i) \neq \varnothing$ by finding a state $X_{[0,T]} \in \mathbb{R}^{\sum_{i=1}^{q}\delta_i \times (T+1)}$ for the data $(u_{[0,T-1]}, y_{[0,T-1]})$. For this, we need a bit of preparation.

Let the subspaces $\mathcal{S}_k \subseteq \mathbb{R}^{1\times k(m+p)}$ with $k \in [1,T]$ be as in Lemma 11.14. Also, denote their dimension by $s_k := \dim \mathcal{S}_k$. For $i \in [1, q+1]$, let $Q_{i,j} \in \mathbb{R}^{s_i \times m}$ and $P_{i,j} \in \mathbb{R}^{s_i\times p}$ with $j \in [1,i]$ be such that the rows of the matrix

$$R_i := \begin{bmatrix} Q_{i,1} & Q_{i,2} & \cdots & Q_{i,i} & P_{i,1} & P_{i,2} & \cdots & P_{i,i}\end{bmatrix}$$

form a basis for $\mathcal{S}_i$. Note that $R_i \in \mathbb{R}^{s_i \times i(m+p)}$.

Since $\mathcal{S}_i \subseteq \operatorname{lker} H_i$ due to (11.39), we have $R_i H_i = 0$ and hence

$$\sum_{j=1}^{i} Q_{i,j} U_{[j-1,T-1-i+j]} + P_{i,j} Y_{[j-1,T-1-i+j]} = 0. \tag{11.42}$$

Due to Lemma 11.14, $\sum_{i=1}^{q+1} s_i = \delta_0 - \delta_{q+1} = p$ since $\delta_0 = p$ and $\delta_{q+1} = 0$ by definition. Therefore,

$$\Pi := \operatorname{col}(P_{1,1}, P_{2,2}, \ldots, P_{q+1,q+1})$$

is a $p \times p$ matrix.

We claim that Π is nonsingular. To see this, let $\eta \in \mathbb{R}^{1\times p}$ be such that $\eta\Pi = 0$. Define

$$R := \begin{bmatrix} 0 & 0 & \cdots & 0 & Q_{1,1} & 0 & 0 & \cdots & 0 & P_{1,1} \\ 0 & 0 & \cdots & Q_{2,1} & Q_{2,2} & 0 & 0 & \cdots & P_{2,1} & P_{2,2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ Q_{q+1,1} & Q_{q+1,2} & \cdots & Q_{q+1,q} & Q_{q+1,q+1} & P_{q+1,1} & P_{q+1,2} & \cdots & P_{q+1,q} & P_{q+1,q+1} \end{bmatrix}$$

and observe that Π is the last block-column of R.

From the definition of $Q_{i,j}$ and $P_{i,j}$, it is straightforward to verify that the rows of R form a basis for the subspace $\bigoplus_{i=1}^{q+1} \sigma^{(q+1-i)}\mathcal{S}_i$. Therefore, $\eta R \in \bigoplus_{i=1}^{q+1} \sigma^{(q+1-i)}\mathcal{S}_i$. This means that $RH_{q+1} = 0$. Since $\eta\Pi = 0$, the last p entries of ηR are zero. Therefore, $\eta R \in (\operatorname{lker} G_{q+1} \times \mathbf{0}_p)$ and hence

$$\eta R \in \left(\bigoplus_{i=1}^{q+1} \sigma^{(q+1-i)}\mathcal{S}_i\right) \cap (\operatorname{lker} G_{q+1} \times \mathbf{0}_p).$$

Since the latter intersection of subspaces is equal to $\{0\}$ due to (11.39), we conclude that $\eta R = 0$. Since the rows of R are linearly independent, we see that $\eta = 0$ and thus $\Pi \in \mathbb{R}^{p\times p}$ is nonsingular.

We now distinguish two cases: $q = 0$ and $q \geqslant 1$.

For the case $q = 0$, we have $s_1 = \delta_0 - \delta_1 = p$ and $\Pi = P_{1,1}$. It follows from (11.42) with $i = 1$ and nonsingularity of $P_{1,1}$ that

$$Y_{[0,T-1]} = -P_{1,1}^{-1} Q_{1,1} U_{[0,T-1]}.$$

Consequently, the memoryless model associated with $-P_{1,1}^{-1} Q_{1,1}$ is consistent with the data, that is $-P_{1,1}^{-1} Q_{1,1} \in \mathcal{E}(0)$. Together with $\sum_{i=1}^{q} \delta_i = 0$, this proves the claim for the case $q = 0$.

For the case $q \geqslant 1$, we first construct some auxiliary sequences from the data and then we show that such sequences can be used to form a state for the data $(u_{[0,T-1]}, y_{[0,T-1]})$.

Let $i \in [2, q+1]$. For $k \in [2, i]$ define $x^{i,k}(0) \in \mathbb{R}^{s_i}$ by

$$x^{i,k}(0) := \sum_{j=k}^{i} (Q_{i,j} u(j-k) + P_{i,j} y(j-k)). \tag{11.43}$$

Define $X_{[1,T]}^{i,k} \in \mathbb{R}^{s_i \times T}$ by

$$X_{[1,T]}^{i,2} := -Q_{i,1} U_{[0,T-1]} - P_{i,1} Y_{[0,T-1]} \tag{11.44}$$

for $k = 2$ and by

$$X_{[1,T]}^{i,k} := X_{[0,T-1]}^{i,k-1} - Q_{i,k-1} U_{[0,T-1]} - P_{i,k-1} Y_{[0,T-1]} \tag{11.45}$$

for $k \in [3, i]$.

Finally, define

$$X_{[0,T]}^{i} := \operatorname{col}(X_{[0,T]}^{i,2}, X_{[0,T]}^{i,3}, \ldots, X_{[0,T]}^{i,i}) \in \mathbb{R}^{(i-1)s_i \times (T+1)}$$

and

$$X_{[0,T]} := \operatorname{col}(X_{[0,T]}^{2}, X_{[0,T]}^{3}, \ldots, X_{[0,T]}^{q+1}) \in \mathbb{R}^{\left(\sum_{i=2}^{q+1} (i-1)s_i\right) \times (T+1)}.$$

Now, we claim that

$$X_{[0,T-1]}^{i,i} = Q_{i,i} U_{[0,T-1]} + P_{i,i} Y_{[0,T-1]}. \tag{11.46}$$

To see this, note first that

$$x^{i,i}(0) = Q_{i,i} u(0) + P_{i,i} y(0) \tag{11.47}$$

due to (11.43). Now, we consider the case $i = 2$. From (11.44), we see that

$$X_{[1,T-1]}^{2,2} = -Q_{2,1} U_{[0,T-2]} - P_{2,1} Y_{[0,T-2]}.$$

In view of (11.42),

$$X^{2,2}_{[1,T-1]} = Q_{2,2}U_{[1,T-1]} + P_{2,2}Y_{[1,T-1]}.$$

Together with (11.47), this shows that (11.46) holds for $i = 2$.

We now prove (11.46) for the case $i > 2$. Let $\alpha \in [1, T-1]$. Consider first the case $\alpha \in [1, i-2]$. Note that

$$\begin{aligned} x^{i,i}(\alpha) - x^{i,i-\alpha}(0) &= \sum_{j=i-\alpha+1}^{i} \left(x^{i,j}(\alpha - i + j) - x^{i,j-1}(\alpha - i + j - 1)\right) \\ &= -\sum_{j=i-\alpha+1}^{i} \left(Q_{i,j-1}u(\alpha - i + j - 1) + P_{i,j-1}y(\alpha - i + j - 1)\right) \end{aligned} \tag{11.48}$$

where the second equality follows from (11.45). Moreover, (11.43) with $k = i-\alpha$ implies that

$$x^{i,i-\alpha}(0) = \sum_{j=i-\alpha}^{i} \left(Q_{i,j}u(j - (i-\alpha)) + P_{i,j}y(j - (i-\alpha))\right).$$

Now, we see from (11.48) that

$$\begin{aligned} x^{i,i}(\alpha) = &\sum_{j=i-\alpha}^{i} \left(Q_{i,j}u(\alpha - i + j) + P_{i,j}y(\alpha - i + j)\right) \\ &-\sum_{j=i-\alpha+1}^{i} \left(Q_{i,j-1}u(\alpha - i + j - 1) + P_{i,j-1}y(\alpha - i + j - 1)\right) \\ &= Q_{i,i}u(\alpha) + P_{i,i}y(\alpha). \end{aligned}$$

Together with (11.47), this proves that

$$X^{i,i}_{[0,i-2]} = Q_{i,i}U_{[0,i-2]} + P_{i,i}Y_{[0,i-2]}. \tag{11.49}$$

Now, consider the case $\alpha \in [i-1, T-1]$. Note that

$$\begin{aligned} x^{i,i}(\alpha) - x^{i,2}(\alpha - i + 2) &= \sum_{k=3}^{i} (x^{i,k}(\alpha - i + k) - x^{i,k-1}(\alpha - i + k - 1)) \\ &= -\sum_{k=3}^{i} (Q_{i,k-1}u(\alpha - i + k - 1) + P_{i,k-1}y(\alpha - i + k - 1)) \end{aligned}$$

where the second equality follows from (11.45). Now, we use (11.44) to obtain

$$x^{i,i}(\alpha) = -\sum_{k=2}^{i}(Q_{i,k-1}u(\alpha-i+k-1) + P_{i,k-1}y(\alpha-i+k-1)),$$

and use (11.42) to conclude that

$$x^{i,i}(\alpha) = Q_{i,i}u(\alpha) + P_{i,i}y(\alpha).$$

Hence, we see that

$$X^{i,i}_{[i-1,T-1]} = Q_{i,i}U_{[i-1,T-1]} + P_{i,i}U_{[i-1,T-1]}.$$

Together with (11.49), this proves that (11.46) holds.

Since Π is nonsingular, $P_{i,i}$ has full row rank. Therefore, (11.46) implies that

$$\operatorname{rsp} Y_{[0,T-1]} \subseteq \operatorname{rsp} \begin{bmatrix} X_{[0,T-1]} \\ U_{[0,T-1]} \end{bmatrix}.$$

Then, it follows from (11.44) and (11.45) that

$$\operatorname{rsp} X_{[1,T]} \subseteq \operatorname{rsp} \begin{bmatrix} X_{[0,T-1]} \\ U_{[0,T-1]} \end{bmatrix}.$$

Therefore, we see from (11.17) that $X_{[0,T]}$ is a state for $(u_{[0,T-1]}, y_{[0,T-1]})$. Note that the number of rows of $X_{[0,T]}$ equals $\sum_{i=2}^{q+1}(i-1)s_i$. Since $s_k = \delta_{k-1} - \delta_k$ due to Lemma 11.14 and since $\delta_{q+1} = 0$, we conclude that $\sum_{i=2}^{q+1}(i-1)s_i = \sum_{i=1}^{q}\delta_i$. Therefore, $\mathcal{E}(\sum_{i=1}^{q}\delta_i) \neq \varnothing$.

To prove the rest, note that $\mathcal{E}(q, \sum_{i=1}^{q}\delta_i) \subseteq \mathcal{E}(\sum_{i=1}^{q}\delta_i)$ by definition. Therefore, it is enough to show that the reverse inclusion holds. To do so, let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{E}(\sum_{i=1}^{q}\delta_i)$ and $\ell = \ell(C, A)$. Suppose that $\ell < q$. Since $T-1 \geqslant q$ by the definition of q in (11.16), we have $T > \ell + 1$. Then, Theorem 11.3.(a) implies that $\ell \geqslant q$. This contradicts $\ell < q$. As such, we conclude that $\ell \geqslant q$. However, Theorem 11.3.(b) implies that $q \geqslant \ell$. Hence, we see that $\ell = q$. This means that $\mathcal{E}(\sum_{i=1}^{q}\delta_i) \subseteq \mathcal{E}(q, \sum_{i=1}^{q}\delta_i)$ which completes the proof. ■

11.5.3 Illustrative example of state construction

To illustrate the constructive proof of Theorem 11.4, we will build a state for the data given in Section 11.3 and for the choices $T = 5$ and $T = 14$.

For $T = 5$, $\delta_0 = \delta_1 = 2$, and $\delta_k = 0$ for $k \in [2, 5]$ (see Table 11.1 in Section 11.3). Then, we see from Lemma 11.14 that $\dim \mathcal{S}_1 = 0$ and $\dim \mathcal{S}_2 = 2$.

Therefore, $P_{1,1}$ and $Q_{1,1}$ are void matrices whereas one can choose the following basis matrix for $\mathcal{S}_2$:

$$\left[\, Q_{2,1} \,\middle|\, Q_{2,2} \,\middle|\, P_{2,1} \,\middle|\, P_{2,2} \,\right] = \left[\begin{array}{cc|cc|cc|cc} -3 & -1 & -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

In view of (11.43)-(11.44), these choices yield the state

$$X_{[0,5]} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

By solving the linear equations (11.9), we obtain the following consistent system:

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] = \left[\begin{array}{cc|cc} -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]. \tag{11.50}$$

Clearly, the data $(u_{[0,4]}, y_{[0,4]})$ are not informative for system identification as there exists a minimal consistent system with two states.

For $T = 14$, $\delta_0 = \delta_1 = 2$, $\delta_2 = 1$, and $\delta_k = 0$ for $k \in [3, 14]$ (see Table 11.1 in Section 11.3). Also, note that the data $(u_{[0,13]}, y_{[0,13]})$ are informative for system identification for all the choices of L_+ and N_+ given in Table 11.2. As such, we can employ Theorem 11.4 to identify an isomorphic system to the true one. To do so, we first observe from Lemma 11.14 that $\dim \mathcal{S}_1 = 0$, $\dim \mathcal{S}_2 = 1$, and $\dim \mathcal{S}_3 = 1$. Therefore, $P_{1,1}$ and $Q_{1,1}$ are void matrices whereas one can choose the following basis matrices for $\mathcal{S}_2$ and $\mathcal{S}_3$:

$$\left[\, Q_{2,1} \,\middle|\, Q_{2,2} \,\middle|\, P_{2,1} \,\middle|\, P_{2,2} \,\right] = \left[\begin{array}{cc|cc|cc|cc} -1 & 0 & -1 & 0 & 0 & -1 & 1 & 0 \end{array}\right].$$

$$\left[\, Q_{3,1} \,\middle|\, Q_{3,2} \,\middle|\, Q_{3,3} \,\middle|\, P_{3,1} \,\middle|\, P_{3,2} \,\middle|\, P_{3,3} \,\right] = \left[\begin{array}{cc|cc|cc|cc|cc|cc} 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

In view of (11.43)-(11.45), these choices yield the state

$$X_{[0,14]} = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 1 & 2 & 2 & 3 & 2 & 2 & 2 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 \end{bmatrix}.$$

By solving the linear equations (11.9), we obtain the following consistent system:

$$\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{ccc|cc} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right].$$

One can easily verify that this system is isomorphic to the true system.

11.6 Proving the sufficiency part of the main result

In this section we prove the sufficiency part of Theorem 11.6. To do so, we need auxiliary results on the ranks of state-input data Hankel matrices.

11.6.1 On the ranks of state-input data Hankel matrices

Let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{E}(n)$ and let $X_{[0,T]}$ be a state for $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Define

$$J_k(X) := \begin{bmatrix} X_{[0,T-k]} \\ H_k(u_{[0,T-1]}) \end{bmatrix}$$

for $k \in [1, T]$. An immediate consequence of the Hankel structure of these matrices is that

$$\operatorname{lker} J_{k-1}(X) \times \mathbf{0}_m \subseteq \operatorname{lker} J_k(X) \tag{11.51}$$

for every $k \in [2, T]$.

Next, we investigate the relationships between ranks of H_k and $J_k(X)$ matrices. Recall from (11.29) that

$$H_k = \Phi_k J_k(X) \quad \text{and} \quad G_k = \Psi_k J_k(X). \tag{11.52}$$

Lemma 11.15. *Suppose that $T \geqslant \ell + 1$. Let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{E}(\ell, n)$ and $X_{[0,T]}$ be a state for $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Then, the following statements hold*

(a) *If (C, A) is observable, then $\operatorname{rank} J_k(X) = \operatorname{rank} H_k$ for all $k \in [d, T]$ where $d = \max(1, \ell)$.*

(b) *If $J_i(X)$ has full row rank for some $i \in [1, T]$, then for each $k \in [1, i]$ $J_k(X)$ has full row rank, $\operatorname{rank} H_k = km + \operatorname{rank} \Omega_k$, and $\operatorname{rank} G_k = km + \operatorname{rank} \Omega_{k-1}$.*

Proof. Statement (a) readily follows from the fact that Φ_k has full column rank for $k \geqslant d$ whenever (C, A) is observable. To prove (b), note first that $J_k(X)$ has full row rank whenever $k \in [1, i]$ due to (11.51). For the rest, observe that (11.29) implies $\operatorname{rank} H_k = \operatorname{rank} \Phi_k$ and $\operatorname{rank} G_k = \operatorname{rank} \Psi_k$ whenever $J_k(X)$ has full row rank. From the definitions, we have $\operatorname{rank} \Phi_k = km + \operatorname{rank} \Omega_k$ and $\operatorname{rank} \Psi_k = km + \operatorname{rank} \Omega_{k-1}$ which completes the proof. □

An interesting and useful consequence of the above lemma is related to the isomorphism property.

Lemma 11.16. *Suppose that $T \geqslant \ell + 1$. Let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{E}(\ell, n) \cap \mathcal{O}$ and $X_{[0,T]}$ be a state for $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. If $J_{\ell+1}(X)$ has full row rank, then $\mathcal{E}(\ell, n) \cap \mathcal{O}$ has the isomorphism property.*

Proof. From Lemma 11.15.(b), we see that $\operatorname{rank} H_{\ell+1} = (\ell+1)m+n$. Let $i \in [1,2]$, $\left[\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right] \in \mathcal{E}(\ell,n) \cap \mathcal{O}$ and let $X^i_{[0,T]} \in \mathbb{R}^{n\times(T+1)}$ be a state for $\left[\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right]$. Also, let J^i_k denote $J_k(X^i)$. Let Ω^i_k, Θ^i_k and Φ^i_k denote the observability, Toeplitz and system matrix in (11.30) of system $\left[\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right]$, respectively. Because of observability of the pair (C_i, A_i), the matrix $\Phi^i_{\ell+1}$ has full column rank. Therefore, it follows from (11.52) that

$$\ker H_{\ell+1} = \ker J^1_{\ell+1} = \ker J^2_{\ell+1}. \tag{11.53}$$

Moreover, it follows from the same equation and the fact that $\operatorname{rank} H_{\ell+1} = (\ell+1)m+n$ that both $J^1_{\ell+1}$ and $J^2_{\ell+1}$ have full row rank. Now, by (11.53), there exist matrices $S \in \mathbb{R}^{n\times n}$, $P \in \mathbb{R}^{n\times(\ell+1)m}$, $Q \in \mathbb{R}^{(\ell+1)m\times n}$ and $R \in \mathbb{R}^{(\ell+1)m\times(\ell+1)m}$ such that

$$\begin{bmatrix} X^2_{[0,T-\ell-1]} \\ H_{\ell+1}(u_{[0,T-1]}) \end{bmatrix} = \begin{bmatrix} S & P \\ Q & R \end{bmatrix} \begin{bmatrix} X^1_{[0,T-\ell-1]} \\ H_{\ell+1}(u_{[0,T-1]}) \end{bmatrix}. \tag{11.54}$$

Obviously, we also have that $H_{\ell+1}(u_{[0,T-1]}) = 0 \cdot X^1_{[0,T-\ell-1]} + I \cdot H_{\ell+1}(u_{[0,T-1]})$, implying that

$$\begin{bmatrix} Q & R-I \end{bmatrix} \begin{bmatrix} X^1_{[0,T-\ell-1]} \\ H_{\ell+1}(u_{[0,T-1]}) \end{bmatrix} = 0.$$

Since $J^1_{\ell+1}$ has full row rank, we conclude that $Q = 0$ and $R = I$. Next, multiplying (11.54) from left by $\Phi^2_{\ell+1}$ yields

$$H_{\ell+1} = \Phi^2_{\ell+1} \begin{bmatrix} X^2_{[0,T-\ell-1]} \\ H_{\ell+1}(u_{[0,T-1]}) \end{bmatrix} = \begin{bmatrix} 0 & I \\ \Omega^2_{\ell+1} S & \Omega^2_{\ell+1} P + \Theta^2_{\ell+1} \end{bmatrix} \begin{bmatrix} X^1_{[0,T-\ell-1]} \\ H_{\ell+1}(u_{[0,T-1]}) \end{bmatrix}.$$

Using the facts that $H_{\ell+1} = \Phi^1_{\ell+1} J^1_{\ell+1}$ and $J^1_{\ell+1}$ has full row rank, we conclude that

$$\begin{bmatrix} 0 & I \\ \Omega^2_{\ell+1} S & \Omega^2_{\ell+1} P + \Theta^2_{\ell+1} \end{bmatrix} = \begin{bmatrix} 0 & I \\ \Omega^1_{\ell+1} & \Theta^1_{\ell+1} \end{bmatrix}. \tag{11.55}$$

In particular, this shows that

$$\Omega^1_{\ell+1} = \Omega^2_{\ell+1} S. \tag{11.56}$$

By observability of the pair (C_1, A_1), it follows from (11.56) that S is nonsingular. By inspection of the first p rows of (11.56), $C_1 = C_2 S$. Moreover, note that (11.56) implies that

$$\Omega^1_\ell A_1 = \Omega^2_\ell A_2 S = \Omega^1_\ell S^{-1} A_2 S.$$

Using the fact that Ω^1_ℓ has full column rank, we obtain $A_1 = S^{-1} A_2 S$. Next, by (11.55) it also follows that

$$\Theta^1_{\ell+1} - \Theta^2_{\ell+1} = \Omega^2_{\ell+1} P. \tag{11.57}$$

Partition

$$P = \begin{bmatrix} P_1 & \cdots & P_\ell & P_{\ell+1} \end{bmatrix},$$

where the matrix P_i has m columns for $i = 1, 2, \ldots, \ell+1$. Recall that $\Theta^i_{\ell+1}$ is a block Toeplitz matrix of the form

$$\Theta^i_{\ell+1} = \begin{bmatrix} \Theta^i_\ell & 0 \\ C_i \Gamma^i_\ell & D_i \end{bmatrix},$$

where Γ^i_ℓ denotes the depth-ℓ controllability matrix of system $\left[\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right]$. As a consequence, the last m columns of $\Theta^1_{\ell+1} - \Theta^2_{\ell+1}$ contain zeros except for (possibly) the last p rows. It thus follows from (11.57) and the fact that Ω^2_ℓ has full column rank that $P_{\ell+1} = 0$. Therefore, $D_1 = D_2$. However, this implies that the last $2m$ columns of $\Theta^1_{\ell+1} - \Theta^2_{\ell+1}$ only contain zeros, except for the last p rows. This shows that also $P_\ell = 0$ and hence $C_1 B_1 = C_2 B_2$. Repeated application of this argument results in $P_i = 0$ for $i = 1, 2, \ldots, \ell+1$ and $C_1 A_1^k B_1 = C_2 A_2^k B_2$ for $k = 0, 1, \ldots, \ell-1$. Finally, the latter inequalities imply that

$$\Omega^1_\ell B_1 = \Omega^2_\ell B_2 = \Omega^1_\ell S^{-1} B_2,$$

which yields $B_1 = S^{-1} B_2$. We conclude that the systems $\left[\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right]$ for $i = 1, 2$ are isomorphic, which completes the proof. □

In view of (11.51), the rank-nullity theorem implies that $\operatorname{rank} J_{k-1}(X) + m \geqslant \operatorname{rank} J_k(X)$ for all $k \in [2, T]$. This relation between the ranks of two consecutive $J_k(X)$ matrices can be related to the controllability of the corresponding consistent system.

Lemma 11.17. *Let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{E}(\ell, n)$ and $X_{[0,T]}$ be a state for $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. If for some $k \in [2, T]$ $\operatorname{rank} J_{k-1}(X) + m = \operatorname{rank} J_k(X)$ and $J_k(X)$ does not have full row rank, then (A, B) is not controllable.*

Proof. Let $k \in [2, T]$ be such that

$$\operatorname{rank} J_{k-1}(X) + m = \operatorname{rank} J_k(X) \tag{11.58}$$

and $J_k(X)$ does not have full row rank. Now, define $\mathcal{A}_k \in \mathbb{R}^{(n+km)\times(n+km)}$ and $\mathcal{B}_k \in \mathbb{R}^{(n+km)\times m}$ by

$$\mathcal{A}_k := \begin{bmatrix} A & B & 0 \\ 0 & 0 & I_{(k-1)m} \\ 0_{m,n} & 0 & 0 \end{bmatrix}, \quad \mathcal{B}_k := \begin{bmatrix} 0 \\ 0 \\ I_m \end{bmatrix}.$$

It can be easily shown using the Kalman controllability matrix of $(\mathcal{A}_k, \mathcal{B}_k)$ that the pair (A, B) is controllable if and only if $(\mathcal{A}_k, \mathcal{B}_k)$ is controllable. Next, the relation $X_{[1,T]} = AX_{[0,T-1]} + BU_{[0,T-1]}$ implies that

$$\mathcal{A}_k J_k = \begin{bmatrix} X_{[1,T-k+1]} \\ H_{k-1}(U_{[1,T-1]}) \\ 0_{m,T-k+1} \end{bmatrix}.$$

Note that the matrix

$$\begin{bmatrix} X_{[1,T-k+1]} \\ H_{k-1}(U_{[1,T-1]}) \end{bmatrix}$$

can be obtained from $J_{k-1}(X)$ by deleting its first column. Hence, we see that

$$\operatorname{im} \mathcal{A}_k J_k(X) \subseteq \operatorname{im} J_{k-1}(X) \times \mathbb{R}^m. \tag{11.59}$$

By taking orthogonal complements on both sides of (11.51), we obtain $\operatorname{im} J_k(X) \subseteq \operatorname{im} J_{k-1}(X) \times \mathbb{R}^m$, and due to (11.58) it holds that

$$\operatorname{im} J_k(X) = \operatorname{im} J_{k-1}(X) \times \mathbb{R}^m. \tag{11.60}$$

Therefore, it follows from (11.59) that $\mathcal{A}_k \operatorname{im} J_k(X) \subseteq \operatorname{im} J_k(X)$, i.e., $\operatorname{im} J_k(X)$ is $\mathcal{A}_k$-invariant. Furthermore, it is evident from (11.60) that $\operatorname{im} \mathcal{B}_k \subseteq \operatorname{im} J_k(X)$. Since the reachable subspace of the pair $(\mathcal{A}_k, \mathcal{B}_k)$ is the *smallest* $\mathcal{A}_k$-invariant subspace containing $\operatorname{im} \mathcal{B}_k$ we see that

$$\operatorname{im} \begin{bmatrix} \mathcal{B}_k & \mathcal{A}_k\mathcal{B}_k & \cdots & \mathcal{A}_k^{n+km-1}\mathcal{B}_k \end{bmatrix} \subseteq \operatorname{im} J_k(X).$$

Since $J_k(X)$ does not have full row rank, the latter inclusion implies that

$$\operatorname{im} \begin{bmatrix} \mathcal{B}_k & \mathcal{A}_k\mathcal{B}_k & \cdots & \mathcal{A}_k^{n+km-1}\mathcal{B}_k \end{bmatrix} \neq \mathbb{R}^{n+km},$$

i.e., the pair $(\mathcal{A}_k, \mathcal{B}_k)$ is not controllable. We conclude that (A, B) is not controllable, which completes the proof. □

11.6.2 Proof of Theorem 11.6: sufficiency part

In view of (11.21), proving sufficiency of Theorem 11.6 requires showing that the conditions (11.22) imply:

(a) $\mathcal{E}_{[L_-,L_+^{\mathrm{a}}],[N_-,N_+]} \cap \mathcal{M} = \mathcal{E}(n_{\mathrm{true}}) \cap \mathcal{M}$, and

(b) $\mathcal{E}(n_{\mathrm{true}}) \cap \mathcal{M}$ has the isomorphism property.

To this end, we need some preparations.

To begin with, it is clear from the definitions of $\ell_{\min}$ and $n_{\min}$ that

$$\mathcal{E} \cap \mathcal{S}(\ell, n) = \varnothing$$

whenever $\ell < \ell_{\min}$ or $n < n_{\min}$. Therefore, we see from (11.22a) and (11.22b) that

$$\mathcal{E}_{[L_-,L_+^{\mathrm{a}}],[N_-,N_+]} = \mathcal{E}_{[\ell_{\min},L_+^{\mathrm{a}}],[n_{\min},N_+]}. \tag{11.61}$$

Next, we compute the ranks of the data Hankel matrices H_k. Let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{E}(n_{\min})$ and let $X_{[0,T]} \in \mathbb{R}^{n_{\min}\times(T+1)}$ be a state for $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Note that

$$\ell_{\min} \leqslant \ell_{\mathrm{true}} \leqslant L_+ \quad \text{and} \quad \ell_{\min} \leqslant L_+^{\mathrm{d}} = N_+ - n_{\min} + \ell_{\min}.$$

As such, we have

$$\ell_{\min} \leqslant L_+^{\mathrm{a}} = \min(L_+, L_+^{\mathrm{d}}).$$

Due to Theorem 11.5, (C, A) is observable. Since $m \geqslant 1$, (11.22c) implies that $T \geqslant L_+^{\mathrm{a}} + 1$. Therefore, it follows from Lemma 11.15.(a) that

$$\operatorname{rank} J_k(X) = \operatorname{rank} H_k$$

for every $k \in [d, L_+^{\mathrm{a}} + 1]$ where $d = \max(1, \ell_{\min})$. In particular, we see from (11.22d) that

$$\operatorname{rank} J_{L_+^{\mathrm{a}}+1}(X) = (L_+^{\mathrm{a}} + 1)m + n_{\min}$$

and hence $J_{L_+^{\mathrm{a}}+1}(X)$ has full row rank. It then follows from Lemma 11.15.(b) that

$$\operatorname{rank} J_k(X) = \operatorname{rank} H_k = km + n_{\min} \tag{11.62}$$

for every $k \in [d, L_+^{\mathrm{a}} + 1]$.

Now, we claim that

$$\mathcal{E}_{[\ell_{\min},L_+^{\mathrm{a}}],[n_{\min},N_+]} \cap \mathcal{M} \subseteq \mathcal{E}(n_{\min}) \cap \mathcal{M}. \tag{11.63}$$

Suppose first that $N_+ = n_{\min}$. Then, (11.63) follows from

$$\mathcal{E}_{[\ell_{\min},L_+^{\mathrm{a}}],[n_{\min},n_{\min}]} \subseteq \mathcal{E}(n_{\min}).$$

Suppose now that $N_+ > n_{\min}$. Let $\ell \in [\ell_{\min}, L_+^{\mathrm{a}}]$, $n \in [n_{\min} + 1, N_+]$, and $\left[\begin{smallmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{smallmatrix}\right] \in \mathcal{E}(\ell, n) \cap \mathcal{O}$. Also, let $\hat{X}_{[0,T]} \in \mathbb{R}^{n\times(T+1)}$ be a state for $\left[\begin{smallmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{smallmatrix}\right]$. As $(\hat{C}, \hat{A})$ is observable and $n \geqslant 1$, we have that $\ell \geqslant 1$. Since $\ell \geqslant \ell_{\min}$, we further see that $\ell \geqslant d$. Then, Lemma 11.15.(a) and (11.62) imply that

$$\operatorname{rank} J_\ell(\hat{X}) + m = \operatorname{rank} J_{\ell+1}(\hat{X}) = (\ell + 1)m + n_{\min}.$$

Since $n > n_{\min}$, $J_{\ell+1}(\hat{X})$ does not have full row rank. Then, Lemma 11.17 implies that $(\hat{A}, \hat{B})$ is not controllable. Therefore, we see that $\mathcal{E}(\ell, n) \cap \mathcal{M} = \varnothing$ whenever $\ell \in [\ell_{\min}, L_+^{\mathrm{a}}]$ and $n \in [n_{\min} + 1, N_+]$. Hence, (11.63) holds.

Note that $\mathcal{E}(\ell_{\text{true}}, n_{\text{true}}) \subseteq \mathcal{E}_{[\ell_{\min}, L_+^{\text{a}}],[n_{\min}, N_+]}$. Then, it follows from (11.63) that

$$\mathcal{E}(\ell_{\text{true}}, n_{\text{true}}) \cap \mathcal{M} \subseteq \mathcal{E}_{[\ell_{\min}, L_+^{\text{a}}],[n_{\min}, N_+]} \cap \mathcal{M}$$
$$\subseteq \mathcal{E}(\ell_{\min}, n_{\min}) \cap \mathcal{M}. \tag{11.64}$$

Therefore, we see that

$$\ell_{\text{true}} = \ell_{\min} \quad \text{and} \quad n_{\text{true}} = n_{\min}, \tag{11.65}$$

proving (11.23a) and (11.23b). Then, we can conclude from (11.64) and Theorem 11.5 that

$$\mathcal{E}(\ell_{\text{true}}, n_{\text{true}}) \cap \mathcal{M} = \mathcal{E}_{[\ell_{\min}, L_+^{\text{a}}],[n_{\min}, N_+]} \cap \mathcal{M}$$
$$= \mathcal{E}(\ell_{\min}, n_{\min}) \cap \mathcal{M} = \mathcal{E}(n_{\min}) \cap \mathcal{M}. \tag{11.66}$$

Thus, condition (a) follows from (11.61), (11.65), and (11.66).

To show (b), note first that (11.62) and Lemma 11.16 imply that $\mathcal{E}(\ell_{\min}, n_{\min}) \cap \mathcal{O}$ has the isomorphism property. Since the true system is minimal, we see that

$$\mathcal{E}(\ell_{\min}, n_{\min}) \cap \mathcal{O} = \mathcal{E}(\ell_{\min}, n_{\min}) \cap \mathcal{M}. \tag{11.67}$$

Then, it follows from (11.66) that (b) holds.

What remains to be proven is (11.23c). To do so, note first that Theorem 11.5 implies that

$$\mathcal{E}(n_{\min}) = \mathcal{E}(\ell_{\min}, n_{\min}) \cap \mathcal{O}.$$

Then, we see from (11.66) and (11.67) that

$$\mathcal{E}(n_{\min}) = \mathcal{E}_{[\ell_{\min}, L_+^{\text{a}}],[n_{\min}, N_+]} \cap \mathcal{M}.$$

Therefore, (11.23c) follows from (11.21) and (11.61). ■

11.7 Proving the necessity part of the main result

To prove the necessity part of Theorem 11.6, we first present four auxiliary lemmas. The first one deals with the construction of a consistent system with n states from another one with n states.

Lemma 11.18. *Suppose that* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{E}(\ell, n) \cap \mathcal{O}$ *for some* $n \geqslant \ell \geqslant 1$. *Let* $X_{[0,T]} \in \mathbb{R}^{n \times (T+1)}$ *be a state for* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. *Also, let* $d = \min(\ell, T-1)$, $\xi \in \mathbb{R}^{1 \times n}$ *and* $\eta_i \in \mathbb{R}^{1 \times m}$ *with* $i \in [0, d]$ *be such that*

$$\xi X_{[0,T-d-1]} + \sum_{i=0}^{d} \eta_i U_{[i,T-d-1+i]} = 0. \tag{11.68}$$

Let $0 \neq \zeta \in \mathbb{R}^n$ *be such that*

$$CA^i\zeta = 0 \quad \text{for} \quad i \in [0, \ell - 2]. \tag{11.69}$$

Define

$$\hat{A} = A + \zeta\xi \qquad \hat{B} = B + E_{-1} \tag{11.70}$$
$$\hat{C} = C \qquad \hat{D} = D + CE_0 \tag{11.71}$$

where E_0 *and* E_{-1} *are determined by the recursion*

$$E_d = 0_{n,m} \quad \text{and} \quad E_{i-1} = \hat{A}E_i + \zeta\eta_i \quad \text{for} \quad i \in [0, d]. \tag{11.72}$$

Then, the following statements hold:

(a) $\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \in \mathcal{E}(\ell, n) \cap \mathcal{O}$.

(b) *If* $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *and* $\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}$ *are isomorphic, then*

  (i) $\eta_i = 0$ *for every* $i \in [0, d]$,

  (ii) $\xi A^i B = 0$ *for every* $i \in [0, n-1]$.

Proof. To prove (a), we first show that there exists $\hat{X}_{[0,T]} \in \mathbb{R}^{n \times (T+1)}$ satisfying

$$\hat{X}_{[1,T]} = \hat{A}\hat{X}_{[0,T-1]} + \hat{B}U_{[0,T-1]} \tag{11.73}$$
$$Y_{[0,T-1]} = \hat{C}\hat{X}_{[0,T-1]} + \hat{D}U_{[0,T-1]}. \tag{11.74}$$

We claim that $\hat{X}_{[0,T]} \in \mathbb{R}^{n \times (T+1)}$ defined by

$$\hat{X}_{[0,T-d]} := X_{[0,T-d]} - \sum_{i=0}^{d-1} E_i U_{[i,T-d+i]} \tag{11.75}$$
$$\hat{x}(i+1) := \hat{A}\hat{x}(i) + \hat{B}u(i) \quad \text{for} \quad i \in [T-d, T-1] \tag{11.76}$$

satisfies (11.73) and (11.74).

To prove this claim, let $k \in [0, T-d-1]$. Note that

$$\hat{x}(k+1) \overset{(11.75)}{=} x(k+1) - \sum_{i=0}^{d-1} E_i u(k+1+i) = Ax(k) + Bu(k) - \sum_{i=0}^{d-1} E_i u(k+1+i)$$

and

$$\hat{A}\hat{x}(k) + \hat{B}u(k) \overset{(11.75)}{=} \hat{A}x(k) + \hat{B}u(k) - \sum_{i=0}^{d-1} \hat{A}E_i u(k+i).$$

Using (11.70) and (11.72), one can verify that the difference between these two expressions is equal to

$$\zeta\xi x(k) + \sum_{i=0}^{d} \zeta\eta_i u(k+i).$$

Therefore, (11.68) implies that

$$\hat{X}_{[1,T-d]} = \hat{A}\hat{X}_{[0,T-d-1]} + \hat{B}U_{[0,T-d-1]}.$$

Together with (11.76), this proves (11.73).

Therefore, it remains to prove (11.74). First, we make a few crucial observations. To begin with, we have

$$CA^{\ell-1}\zeta \neq 0 \tag{11.77}$$

since (C, A) is observable and $\zeta \neq 0$. Also, it follows from (11.69) and (11.70) that

$$C\hat{A}^i = CA^i \text{ for } i \in [0, \ell-1] \tag{11.78}$$

and $C\hat{A}^\ell = CA^\ell + CA^{\ell-1}\zeta\xi$. Further, observe that

$$C\hat{A}^i\zeta = 0 \tag{11.79}$$

for $i \in [0, \ell-2]$ due to (11.69) and (11.78). Finally, it follows from the recursion (11.72) that

$$E_i = \sum_{k=0}^{d-i-1} \hat{A}^k\zeta\eta_{i+k+1} \tag{11.80}$$

for $i \in [-1, d]$ and from (11.79) that

$$CE_i = 0 \text{ for } i \in [1, d]. \tag{11.81}$$

To show (11.74), we first deal with the case $d = 0$. Since $d = \min(\ell, T-1)$ and $\ell \geqslant 1$, we see that $T = 1$ in this case. Then, it follows from (11.75) that $\hat{X}_{[0,1]} = X_{[0,1]}$ and from (11.71)-(11.72) that $\hat{D} = D$. Since $\hat{C} = C$ due to (11.71), we see that (11.74) is readily satisfied if $d = 0$.

Suppose now that $d \geqslant 1$. Note that

$$\begin{aligned}
\hat{C}\hat{X}_{[0,T-d]} + \hat{D}U_{[0,T-d]} &\overset{(11.71)}{=} C\hat{X}_{[0,T-d]} + \hat{D}U_{[0,T-d]} \\
&\overset{(11.75)\&(11.81)}{=} CX_{[0,T-d]} - CE_0U_{[0,T-d]} + \hat{D}U_{[0,T-d]} \\
&\overset{(11.71)}{=} CX_{[0,T-d]} + DU_{[0,T-d]} = Y_{[0,T-d]}
\end{aligned} \tag{11.82}$$

where the last equality follows from the fact that X is a state for the consistent system $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Hence, we see that (11.74) is satisfied if $d = 1$.

Suppose that $d \geqslant 2$. In view of (11.82), what remains to be proven is that

$$Y_{[T-d+1,T-1]} = \hat{C}\hat{X}_{[T-d+1,T-1]} + \hat{D}U_{[T-d+1,T-1]}. \tag{11.83}$$

To do so, let $i \in [1, d-1]$. Define $\hat{y}(T-d+i) := \hat{C}\hat{x}(T-d+i) + \hat{D}u(T-d+i)$ and $\Delta(T-d+i) := \hat{y}(T-d+i) - y(T-d+i)$. Note that

$$\hat{x}(T-d+i) = \hat{A}^i\hat{x}(T-d) + \sum_{j=0}^{i-1} \hat{A}^{i-j-1}\hat{B}u(T-d+j)$$

$$x(T-d+i) = A^i x(T-d) + \sum_{j=0}^{i-1} A^{i-j-1}Bu(T-d+j)$$

$$\hat{x}(T-d) = x(T-d) - \sum_{j=0}^{d-1} E_j u(T-d+j)$$

where the first equality follows from (11.76), the second from the fact that X is a state for the data, and the third from (11.75). By using (11.70), (11.72), (11.78), (11.79), and the fact that $d \leqslant \ell$, we see that

$$\Delta(T-d+i) = \sum_{j=0}^{i} C\hat{A}^{i-j}E_0 u(T-d+j) - \sum_{j=0}^{d-1} C\hat{A}^i E_j u(T-d+j).$$

By using (11.79) and (11.80), one can prove by induction that

$$C\hat{A}^i E_j = C\hat{A}^{i-j}E_0 \tag{11.84}$$

for all $j \in [0, d-1]$ and $i \in [j, d-1]$ as well as that

$$C\hat{A}^i E_j = 0 \tag{11.85}$$

for all $j \in [1, d-1]$ and $i \in [0, j-1]$. It follows from (11.84) that

$$\sum_{j=0}^{i} C\hat{A}^i E_j u(T-d+j) = \sum_{j=0}^{i} C\hat{A}^{i-j}E_0 u(T-d+j).$$

Hence, we have

$$\Delta(T-d+i) = -\sum_{j=i+1}^{d-1} C\hat{A}^i E_j u(T-d+j) \overset{(11.85)}{=} 0.$$

This proves (11.83) and hence (11.76) in view of (11.82). Therefore, we proved that $\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \in \mathcal{E}(n)$. Further, it follows from (11.78) and observability of (C, A) that $(\hat{C}, \hat{A})$ is also observable and $\ell(\hat{C}, \hat{A}) = \ell$. Then, we have $\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \in \mathcal{E}(\ell, n) \cap \mathcal{O}$ which proves (a).

To prove (b), note that

$$CE_0 = 0 \quad \text{and} \quad CA^i B = \hat{C}\hat{A}^i\hat{B} \quad \text{for all} \quad i \geqslant 0 \tag{11.86}$$

since the two systems are isomorphic. The latter, together with (11.78), implies that

$$CA^i E_{-1} = 0$$

for all $i \in [0, \ell-1]$. As (C, A) is observable and $\ell = \ell(C, A)$, we see that $E_{-1} = 0$. Since $E_{-1} = \hat{A}E_0 + \zeta\eta_0$, (11.77) and (11.78) imply that

$$0 = CA^i E_{-1} = CA^i(\hat{A}E_0 + \zeta\eta_0) = CA^{i+1}E_0 \tag{11.87}$$

for all $i \in [0, \ell-2]$. As (C, A) is observable and $\ell = \ell(C, A)$, (11.86) and (11.87) imply that $E_0 = 0$. Therefore, we have $\zeta\eta_0 = E_{-1} - \hat{A}E_0 = 0$. Since $\zeta \neq 0$, this yields $\eta_0 = 0$. Note that

$$E_0 = \sum_{k=0}^{\ell-1} \hat{A}^k \zeta \eta_{k+1} = 0$$

due to (11.80). From (11.77) and (11.78), we have $C\hat{A}^{\ell-1}\zeta \neq 0$. As such, the vectors $\hat{A}^i\zeta$ with $i \in [0, \ell-1]$ are linearly independent. Then, it follows from $\sum_{k=0}^{\ell-1} \hat{A}^k\zeta\eta_{k+1} = 0$ that $\eta_i = 0$ for every $i \in [1, \ell-1]$. Thus, we have proven (i).

To prove (ii), note first that $\hat{B} = B$ as $E_{-1} = 0$. Then, we have

$$0 = C(sI - \hat{A})^{-1}B - C(sI - A)^{-1}B = C(sI - \hat{A})^{-1}\zeta\xi(sI - A)^{-1}B$$

where the first equality follows from isomorphism, the second is evident. Since $C(sI - \hat{A})^{-1}\zeta$ is a nonzero column vector, we see that $\xi(sI - A)^{-1}B = 0$. This proves (ii). □

Example 11.19. To illustrate Lemma 11.18 by an example, consider the data given in Section 11.3. Also, consider the consistent system (11.50) for the data $(u_{[0,4]}, y_{[0,4]})$. Note that (11.68) is satisfied with $\xi = \begin{bmatrix} -1 & -1 \end{bmatrix}$, $\eta_0 = \begin{bmatrix} 0 & 1 \end{bmatrix}$, and $\eta_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Since the lag of the system (11.50) is 1, the choice $\zeta = \mathrm{col}(1, 1)$ satisfies (11.69). By applying Lemma 11.18, we see that the system

$$\left[\begin{array}{c|c} \hat{A}_1 & \hat{B}_1 \\ \hline \hat{C}_1 & \hat{D}_1 \end{array}\right] = \left[\begin{array}{cc|cc} -2 & -1 & -2 & 2 \\ -1 & -1 & -2 & 2 \\ \hline 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

is consistent with the data with the state sequence

$$X_{[0,5]} = \begin{bmatrix} -1 & 0 & -1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 0 & 1 & 1 \end{bmatrix}.$$

■

Next, the second auxiliary lemma presents a necessary condition for the isomorphism property to hold.

Lemma 11.20. *Suppose that $n \geqslant \ell \geqslant 0$ and $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{E}(\ell, n) \cap \mathcal{M}$. Let $X_{[0,T]} \in \mathbb{R}^{n\times(T+1)}$ be a state for $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. If $\mathcal{E}(\ell, n) \cap \mathcal{M}$ has the isomorphism property, then $T \geqslant \ell + (\ell+1)m + n$ and $J_{\ell+1}(X)$ has full row rank.*

Proof. Suppose, first, that $\ell = n = 0$. Since $\mathcal{E}(0,0) \cap \mathcal{M}$ has the isomorphism property, $U_{[0,T-1]} = J_1(X)$ must have full row rank and hence $T \geqslant m$.

Now, suppose that $n \geqslant \ell \geqslant 1$. Let $d = \min(\ell, T-1)$, $\xi \in \mathbb{R}^{1\times n}$ and $\eta_i \in \mathbb{R}^{1\times m}$ with $i \in [0,d]$ be vectors such that

$$\begin{bmatrix} \xi & \eta_0 & \cdots & \eta_d \end{bmatrix} \in \operatorname{lker} J_{d+1}(X).$$

Also, let ζ_0 be a nonzero vector be such that $CA^i\zeta_0 = 0$ for $i \in [0, \ell-2]$. For $\varepsilon > 0$, let $\left[\begin{smallmatrix} \hat{A}_\varepsilon & \hat{B}_\varepsilon \\ \hat{C}_\varepsilon & \hat{D}_\varepsilon \end{smallmatrix}\right]$ denote the consistent system obtained from Lemma 11.18 by taking $\zeta = \varepsilon\zeta_0$. Since (A, B) is controllable, so is $(\hat{A}_\varepsilon, \hat{B}_\varepsilon)$ for all sufficiently small ε. Hence, we see that

$$\begin{bmatrix} \hat{A}_\varepsilon & \hat{B}_\varepsilon \\ \hat{C}_\varepsilon & \hat{D}_\varepsilon \end{bmatrix} \in \mathcal{E}(\ell, n) \cap \mathcal{M}$$

for some $\varepsilon > 0$. Since $\mathcal{E}(\ell, n) \cap \mathcal{M}$ has the isomorphism property by assumption and (A, B) is controllable, Lemma 11.18.(b) implies that $\xi = 0$ and $\eta_i = 0$ for all $i \in [0,d]$. This means that $J_{d+1}(X)$ has full row rank. Since $T \geqslant 1$, $m \geqslant 1$, and $n \geqslant 1$, $J_T(X)$ has at least 2 rows and exactly 1 column. As such, it cannot have full row rank. Then, we see that $d = \min(\ell, T-1) \neq T-1$. Therefore, $d = \ell$ and $J_{\ell+1}(X)$ has full row rank. The latter implies that $T \geqslant \ell + (\ell+1)m + n$. □

The third auxiliary lemma introduces a way of extending an observable state-space system while preserving observability.

Lemma 11.21. *Suppose that $n \geqslant 1$, $A \in \mathbb{R}^{n\times n}$, and $C \in \mathbb{R}^{p\times n}$ are such that (C, A) is observable. Denote $\ell = \ell(C, A)$. Let $\zeta \in \mathbb{R}^n$ be such that*

$$CA^i\zeta = 0 \quad \forall\, i \in [0, \ell-2] \quad \text{and} \quad CA^{\ell-1}\zeta \neq 0. \tag{11.88}$$

Also, let $n' \geqslant 1$, $A' \in \mathbb{R}^{n' \times n'}$ and $C' \in \mathbb{R}^{1\times n'}$ be such that (C', A') is observable. Then, the pair

$$(\bar{C}, \bar{A}) := \left(\begin{bmatrix} C & 0_{p\times n'} \end{bmatrix}, \begin{bmatrix} A & \zeta C' \\ 0 & A' \end{bmatrix} \right)$$

is observable and $\bar{\ell} := \ell(\bar{C}, \bar{A}) = \ell + n'$. Moreover, if $\zeta' \in \mathbb{R}^{n'}$ satisfies

$$C'(A')^i \zeta' = 0 \quad \forall\, i \in [0, n'-2] \quad \textit{and} \quad C'(A')^{n'-1}\zeta' \neq 0, \tag{11.89}$$

then

$$\bar{C}\bar{A}^i \begin{bmatrix} 0 \\ \zeta' \end{bmatrix} = 0 \quad \forall\, i \in [0, \bar{\ell}-2] \quad \textit{and} \quad \bar{C}\bar{A}^{\bar{\ell}-1} \begin{bmatrix} 0 \\ \zeta' \end{bmatrix} \neq 0. \tag{11.90}$$

Proof. By direct inspection, we see that $\bar{C}\bar{A}^k = \begin{bmatrix} CA^k & \Xi_k \end{bmatrix}$ where $\Xi_0 = 0$, $\Xi_{k+1} = CA^k \zeta C' + \Xi_k A'$ for all $k \geqslant 0$. By using (11.88), we further see that

(i) $\Xi_k = 0$ for all $k \in [0, \ell - 1]$, and

(ii) $\Xi_\ell = CA^{\ell-1}\zeta C'$.

Let $\bar{\Omega}_k$, Ω_k, and Ω'_k denote the k-th observability matrices of the pairs $(\bar{C}, \bar{A})$, (C, A), and (C', A'), respectively. We claim that

$$\operatorname{rank} \bar{\Omega}_{\ell+i} = n + \operatorname{rank} \Omega'_i \tag{11.91}$$

for all $i \geqslant 1$. To show this, let $i \geqslant 1$. Note that $\bar{\Omega}_{\ell+i}$ is of the form

$$\bar{\Omega}_{\ell+i} = \begin{bmatrix} \Omega_\ell & 0 \\ * & \Xi_\ell \\ * & \Xi_{\ell+1} \\ \vdots & \vdots \\ * & \Xi_{\ell+i-1} \end{bmatrix}. \tag{11.92}$$

From (ii) and (11.88), it follows that

$$\ker \begin{bmatrix} \Xi_\ell \\ \Xi_{\ell+1} \\ \vdots \\ \Xi_{\ell+i-1} \end{bmatrix} = \ker \Omega'_i \text{ and } \operatorname{rank} \begin{bmatrix} \Xi_\ell \\ \Xi_{\ell+1} \\ \vdots \\ \Xi_{\ell+i-1} \end{bmatrix} = \operatorname{rank} \Omega'_i. \tag{11.93}$$

Since (C, A) is observable and $\ell(C, A) = \ell$, $\operatorname{rank} \Omega_\ell = n$. Therefore, we see from (11.92) that (11.91) holds. Since $C' \in \mathbb{R}^{1\times n'}$ and (C', A') is observable, we have $\ell(C', A') = n'$. Then, (11.91) implies that $\operatorname{rank} \bar{\Omega}_{\ell+n'} = n + n'$ and hence that $(\bar{C}, \bar{A})$ is observable. It also follows from (11.91) that $\operatorname{rank} \bar{\Omega}_{\ell+n'-1} < n+n'$. This means that $\ell(\bar{C}, \bar{A}) = \ell + n'$. Further, if ζ' satisfies (11.89) then $\zeta' \in \ker \Omega'_{n'-1}$ and $\zeta' \notin \ker \Omega'_{n'}$. From (11.92) and (11.93), it follows that $\begin{bmatrix} 0 \\ \zeta' \end{bmatrix} \in \ker \bar{\Omega}_{\ell+n'-1}$ and $\begin{bmatrix} 0 \\ \zeta' \end{bmatrix} \notin \ker \bar{\Omega}_{\ell+n'}$. Hence, (11.90) holds. □

The final auxiliary lemma presents a condition under which the matrix $J_k(X)$ has full row rank for a certain depth k.

Lemma 11.22. *Suppose that $n \geqslant \ell \geqslant 0$ and $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{E}(\ell, n) \cap \mathcal{M}$. Let $X_{[0,T]} \in \mathbb{R}^{n\times(T+1)}$ be a state for $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. If $\mu \geqslant 1$ and $\mathcal{E}(\ell+\mu, n+\mu) \cap \mathcal{M} = \varnothing$, then $T \geqslant \ell + \mu + (\ell+\mu+1)m + n$ and $J_{\ell+\mu+1}(X)$ has full row rank.*

Proof. Let $\lambda \in \mathbb{R}$, $A'_\lambda \in \mathbb{R}^{\mu\times\mu}$ be the Jordan block with the eigenvalue λ, i.e.

$$A'_\lambda = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix},$$

$C' := e_1^T$, and $\zeta' := e_\mu$ where e_i denotes the ith standard basis vector of $\mathbb{R}^\mu$. Clearly, (C', A'_λ) is observable and $\ell(C', A'_\lambda) = \mu$. In addition, we have that

$$C'(A'_\lambda)^i\zeta' = 0 \quad \forall\, i \in [0, \mu-2] \quad \text{and} \quad C'(A'_\lambda)^{\mu-1}\zeta' = 1.$$

We claim that $J_{d+1}(x)$ has full row rank where $d = \min(\ell+\mu, T-1)$. To prove this claim, we distinguish two cases: $n = 0$ and $n \geqslant 1$.

For the case $n = 0$, we have that $\ell = 0$, $X_{[0,T]}$ is a void matrix, and $Y_{[0,T-1]} = DU_{[0,T-1]}$ for some $D \in \mathbb{R}^{p\times m}$. Therefore, for every nonzero $\theta \in \mathbb{R}^p$, $Z_{[0,T]} := 0_{\mu\times(T+1)}$ is a state for

$$\begin{bmatrix} A'_\lambda & 0_{\mu,m} \\ \theta C' & D \end{bmatrix} \in \mathcal{E}(\mu,\mu) \cap \mathcal{O}.$$

Let η_i with $i \in [0, d]$ be such that $\begin{bmatrix} \eta_0 & \cdots & \eta_d \end{bmatrix} \in \operatorname{lker} J_{d+1}(X)$. Clearly, we have $\begin{bmatrix} 0_{1,\mu} & \eta_0 & \cdots & \eta_d \end{bmatrix} \in \operatorname{lker} J_{d+1}(Z)$. Define

$$\hat{A}_\lambda = A'_\lambda, \quad \hat{B}_\lambda = E_{-1}, \quad \hat{C} = \theta C', \quad \text{and} \quad \hat{D}_\lambda = D + \theta C' E_0$$

where $E_d = 0$ and $E_{i-1} = \hat{A}_\lambda E_i + \zeta'\eta_i$ for $i \in [0, d]$. Since $d = \min(\mu, T-1)$ for this case, it follows from Lemma 11.18.(a) that

$$\begin{bmatrix} \hat{A}_\lambda & \hat{B}_\lambda \\ \hat{C} & \hat{D}_\lambda \end{bmatrix} \in \mathcal{E}(\mu,\mu) \cap \mathcal{O}.$$

Since $\mathcal{E}(\mu,\mu) \cap \mathcal{M} = \varnothing$ due to the hypothesis, $(\hat{A}_\lambda, \hat{B}_\lambda)$ is uncontrollable. From the fact that $\hat{A}_\lambda = A'_\lambda$ is a Jordan block, we see that $(\zeta')^T A'_\lambda = \lambda(\zeta')^T$ and $(\zeta')^T E_{-1} = 0$. Since

$$E_{-1} = \sum_{k=0}^{d} (A'_\lambda)^k \zeta' \eta_k$$

due to (11.80), we see that $\sum_{k=0}^{d} \lambda^k \eta_k = 0$. As λ is an arbitrary real number, we conclude that $\eta_i = 0$ for every $i \in [0, d]$ and hence $J_{d+1}(X)$ has full row rank.

For the case $n \geqslant 1$, let $\zeta \in \mathbb{R}^n$ be as in (11.88) and define

$$\bar{C} := \begin{bmatrix} C & 0_{p\times\mu} \end{bmatrix}, \quad \bar{A}_{\varepsilon,\lambda} := \begin{bmatrix} A & \varepsilon\zeta C' \\ 0 & A'_\lambda \end{bmatrix}, \quad \text{and} \quad \bar{B} := \begin{bmatrix} B \\ 0_{\mu\times m} \end{bmatrix}$$

for $\varepsilon > 0$. Then, it follows from Lemma 11.21 that $(\bar{C}, \bar{A}_{\varepsilon,\lambda})$ is observable, $\ell(\bar{C}, \bar{A}_{\varepsilon,\lambda}) = \ell + \mu =: \bar{\ell}$,

$$\bar{C}\bar{A}^i_{\varepsilon,\lambda} \begin{bmatrix} 0 \\ \zeta' \end{bmatrix} = 0 \quad \forall\, i \in [0, \bar{\ell} - 2] \quad \text{and} \quad \bar{C}\bar{A}^{\bar{\ell}-1}_{\varepsilon,\lambda} \begin{bmatrix} 0 \\ \zeta' \end{bmatrix} \neq 0. \tag{11.94}$$

Note that

$$\begin{bmatrix} \bar{A}_{\varepsilon,\lambda} & \bar{B} \\ \bar{C} & D \end{bmatrix} \in \mathcal{E}(\ell + \mu, n + \mu) \cap \mathcal{O}$$

and

$$Z_{[0,T]} := \begin{bmatrix} X_{[0,T]} \\ 0_{\mu\times(T+1)} \end{bmatrix}$$

is a state for $\begin{bmatrix} \bar{A}_{\varepsilon,\lambda} & \bar{B} \\ \bar{C} & D \end{bmatrix}$. Let $d = \min(\ell + \mu, T - 1)$. Also, let $\xi \in \mathbb{R}^{1\times n}$, η_i with $i \in [0, d]$ be such that $\begin{bmatrix} \xi & \eta_0 & \cdots & \eta_d \end{bmatrix} \in \operatorname{lker} J_{d+1}(X)$. Clearly, we have $\begin{bmatrix} \xi & 0_{1\times\mu} & \eta_0 & \cdots & \eta_d \end{bmatrix} \in \operatorname{lker} J_{d+1}(Z)$. Define

$$\begin{aligned} \hat{A}_{\varepsilon,\lambda} &= \bar{A}_{\varepsilon,\lambda} + \begin{bmatrix} 0 \\ \zeta' \end{bmatrix} \begin{bmatrix} \xi & 0 \end{bmatrix} & \hat{B}_{\varepsilon,\lambda} &= \bar{B} + E_{-1} \\ \hat{C} &= \bar{C} & \hat{D}_{\varepsilon,\lambda} &= D + \bar{C}E_0 \end{aligned}$$

where

$$E_d = 0 \quad \text{and} \quad E_{i-1} = \hat{A}_{\varepsilon,\lambda} E_i + \begin{bmatrix} 0 \\ \zeta' \end{bmatrix} \eta_i \ \text{ for } \ i \in [0, d].$$

Then, it follows from Lemma 11.18.(a) that

$$\begin{bmatrix} \hat{A}_{\varepsilon,\lambda} & \hat{B}_{\varepsilon,\lambda} \\ \hat{C} & \hat{D}_{\varepsilon,\lambda} \end{bmatrix} \in \mathcal{E}(\ell + \mu, n + \mu) \cap \mathcal{O}.$$

From the hypothesis, we know that $(\hat{A}_{\varepsilon,\lambda}, \hat{B}_{\varepsilon,\lambda})$ is uncontrollable. By taking the limit as ε tends to zero, we conclude that $(\hat{A}_{0,\lambda}, \hat{B}_{0,\lambda})$ is uncontrollable as well. Note that

$$\hat{A}_{0,\lambda} = \begin{bmatrix} A & 0 \\ \zeta'\xi & A'_\lambda \end{bmatrix} \quad \text{and} \quad \hat{B}_{0,\lambda} = \begin{bmatrix} B \\ 0_{\mu\times m} \end{bmatrix} + F_{-1}$$

where

$$F_d = 0 \quad \text{and} \quad F_{i-1} = \hat{A}_{0,\lambda} F_i + \begin{bmatrix} 0 \\ \zeta' \end{bmatrix} \eta_i \text{ for } i \in [0, d].$$

Suppose that λ is not an eigenvalue of A. Then, every left eigenvector of $\hat{A}_{0,\lambda}$ corresponding to an eigenvalue of A must be of the form $\begin{bmatrix} v & 0 \end{bmatrix}$ where $v \in \mathbb{C}^{1\times n}$. From

$$(\zeta')^T A'_\lambda = \lambda(\zeta')^T,$$

we see that left eigenvectors of $\hat{A}_{0,\lambda}$ corresponding to the eigenvalue λ are nonzero multiples of $\begin{bmatrix} \xi(\lambda I - A)^{-1} & (\zeta')^T \end{bmatrix}$. Since (A, B) is controllable but $(\hat{A}_{0,\lambda}, \hat{B}_{0,\lambda})$ is uncontrollable, it follows from the Hautus test that

$$\begin{bmatrix} \xi(\lambda I - A)^{-1} & (\zeta')^T \end{bmatrix} \hat{B}_{0,\lambda} = 0.$$

Since $F_{-1} = \sum_{k=0}^{d} \hat{A}_{0,\lambda}^k \begin{bmatrix} 0 \\ \zeta' \end{bmatrix} \eta_k$, we see that $\xi(\lambda I - A)^{-1}B + \sum_{k=0}^{d} \lambda^k \eta_k = 0$. As this equality should hold for all $\lambda \in \mathbb{R}$ that is not an eigenvalue of A, we can conclude that $\eta_i = 0$ for $i \in [0, d]$ and $\xi(\lambda I - A)^{-1}B = 0$. The latter implies that $\xi = 0$ since (A, B) is controllable. Consequently, $J_{d+1}(X)$ has full row rank.

To prove that $J_{\ell+\mu+1}(X)$ has full row rank, note that $J_T(X)$ has at least 2 rows and exactly 1 column since $T \geqslant 1$, $m \geqslant 1$, and $n \geqslant 1$. As such, it cannot have full row rank. Then, we see that $d = \min(\ell + \mu, T - 1) \neq T - 1$. Therefore, $d = \ell + \mu$ and thus $J_{\ell+\mu+1}(X)$ has full row rank. The latter implies that $T \geqslant \ell + \mu + (\ell + \mu + 1)m + n$. □

11.7.1 Proof of Theorem 11.6: necessity part

Suppose that the data are informative for system identification within

$$\mathcal{S}_{[L_-,L_+],[N_-,N_+]} \cap \mathcal{M}.$$

Let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{E}(\ell_{\text{true}}, n_{\text{true}}) \cap \mathcal{M}$ and let $X_{[0,T]} \in \mathbb{R}^{n\times(T+1)}$ be a state for $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Since $\mathcal{E}(n_{\text{true}}) \cap \mathcal{M}$ has the isomorphism property, we have $\mathcal{E}(n_{\text{true}}) \cap \mathcal{M} = \mathcal{E}(\ell_{\text{true}}, n_{\text{true}}) \cap \mathcal{M}$. Then, Lemma 11.20 implies that

$$T \geqslant \ell_{\text{true}} + (\ell_{\text{true}} + 1)m + n_{\text{true}} \tag{11.95}$$

and $J_{\ell_{\text{true}}+1}(X)$ has full row rank whereas Lemma 11.15.(b) implies that $\delta_k = \rho_k$ for every $k \in [0, \ell_{\text{true}} + 1]$. As $\rho_{\ell_{\text{true}}} \geqslant 1$ due to Lemma 11.12.(b), we see that $q \geqslant \ell_{\text{true}}$. Then, Theorem 11.5 implies that $\ell_{\min} \geqslant \ell_{\text{true}}$. Since the reverse inequality readily follows from the definition of $\ell_{\min}$ in (11.18), we have $\ell_{\text{true}} = \ell_{\min}$. Further, Theorem 11.5 and Lemma 11.12.(c) imply that $n_{\text{true}} = n_{\min}$. Then, (11.22a) and (11.22b) follow from (11.10).

If $L_+^{\mathrm{a}} = \ell_{\mathrm{true}}$, (11.22c) readily follows from (11.95). As such, (11.22d) follows from $\operatorname{rank} H_{\ell_{\mathrm{true}}+1} = \operatorname{rank} J_{\ell_{\mathrm{true}}+1}(X)$ and $J_{\ell_{\mathrm{true}}+1}(X)$ having full row rank. Suppose that $L_+^{\mathrm{a}} > \ell_{\mathrm{true}}$. Note that the informativity of the data for system identification within $\mathcal{S}_{[L_-,L_+],[N_-,N_+]} \cap \mathcal{M}$ implies that $\mathcal{E}(\ell_{\mathrm{true}} + \mu, n_{\mathrm{true}} + \mu) \cap \mathcal{M} = \varnothing$ where $\mu = L_+^{\mathrm{a}} - \ell_{\mathrm{true}}$. Then, Lemma 11.22 implies that (11.22c) holds and $J_{L_+^{\mathrm{a}}+1}(X)$ has full row rank. Since $\operatorname{rank} H_{L_+^{\mathrm{a}}+1} = \operatorname{rank} J_{L_+^{\mathrm{a}}+1}(X)$, we see that (11.22d) holds. ■

11.8 A simple proof of the fundamental lemma

In this section we use the machinery developed in this chapter to prove the fundamental lemma, i.e., Theorem 1.2. We will also prove Proposition 11.2 (c).

11.8.1 Proof of Theorem 1.2

We assume that $(A_{\mathrm{true}}, B_{\mathrm{true}})$ is controllable and $u_{[0,T-1]}$ is persistently exciting of order $N + L$. We start by proving item (a). In other words, we want to show that the matrix $J_L(X)$ has full row rank.

Suppose on the contrary that $\operatorname{rank} J_L(X) < n_{\mathrm{true}} + mL$. Obviously, by the persistency of excitation condition, $\operatorname{rank} J_{N+L}(X) \geqslant (N + L)m$. The rank difference between $J_L(X)$ and $J_{N+L}(X)$ is thus at least $(m - 1)N + 1$. Hence, there exists a $k \in \{L + 1, L + 2, \ldots, N + L\}$ such that $\operatorname{rank} J_k(X) = m + \operatorname{rank} J_{k-1}(X)$.

By Lemma 11.15 (b) and the fact that $J_L(X)$ does not have full row rank, $J_k(X)$ does not have full row rank. Therefore, it follows from Lemma 11.17 that $(A_{\mathrm{true}}, B_{\mathrm{true}})$ is uncontrollable. This is a contradiction. Hence, we conclude that $\operatorname{rank} J_L(X) = n_{\mathrm{true}} + mL$ which proves item (a).

Next, we prove (b). The 'if' part readily follows from the discussion following Equation (1.2). To prove the 'only if' part, let $(\bar{u}_{[0,L-1]}, \bar{y}_{[0,L-1]})$ be a restricted input-output trajectory on the time interval $[0, L - 1]$. Let $\bar{x}(0)$ be an initial state of (1.1) compatible with this input-output trajectory. By item (a), the matrix $J_L(X)$ has full row rank. Therefore, there exists a vector $g \in \mathbb{R}^{T-L+1}$ such that

$$\begin{bmatrix} \bar{x}(0) \\ \bar{u}_{[0,L-1]} \end{bmatrix} = J_L(X)g. \tag{11.96}$$

Let Ω_L and Θ_L be the observability and system matrices of the true system, and define Φ_L as in (11.30). By multiplying both sides of (11.96) by the matrix Φ_L we obtain

$$\begin{bmatrix} \bar{u}_{[0,L-1]} \\ \bar{y}_{[0,L-1]} \end{bmatrix} = \begin{bmatrix} H_L(u_{[0,T-1]}) \\ H_L(y_{[0,T-1]}) \end{bmatrix} g,$$

which proves (b).

Finally, we prove (c). Let $i \in \mathbb{Z}_+$. In view of item (b), it suffices to show that the controllability of $(A_{\text{true}}, B_{\text{true}})$ implies that the spaces of restricted input-output trajectories of (1.1) on the intervals $[0, L-1]$ and $[i, i+L-1]$ coincide. In other words, it is sufficient to prove the following claim.

Claim: Let

$$v = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{L-1} \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{L-1} \end{bmatrix},$$

where $v_i \in \mathbb{R}^m$ and $z_i \in \mathbb{R}^p$ for $i = 0, 1, \ldots, L-1$. Then (v, z) is a restricted input-output trajectory of (1.1) on the interval $[0, L-1]$ if and only if it is a restricted input-output trajectory of (1.1) on the interval $[i, i+L-1]$.

To prove this claim, note that the 'if' statement follows directly from time-invariance of (1.1). Therefore, we focus on proving the 'only if' statement. We will first show that the matrix

$$\begin{bmatrix} A_{\text{true}}^i & A_{\text{true}}^{i-1} B_{\text{true}} & \cdots & A_{\text{true}} B_{\text{true}} & B_{\text{true}} \end{bmatrix} \tag{11.97}$$

has full row rank. Let $\xi \in \mathbb{R}^{1 \times n}$ be such that

$$\xi \begin{bmatrix} A_{\text{true}}^i & A_{\text{true}}^{i-1} B_{\text{true}} & \cdots & A_{\text{true}} B_{\text{true}} & B_{\text{true}} \end{bmatrix} = 0.$$

Then $\xi A_{\text{true}}^k B_{\text{true}} = 0$ for all $k \in \mathbb{Z}_+$. By controllability of the pair $(A_{\text{true}}, B_{\text{true}})$, it follows that $\xi = 0$. Therefore, (11.97) has full row rank. Now, let

$$(u(t), x(t), y(t))_{t=0}^{L-1}$$

be a restricted input-state-output trajectory of (1.1) such that $u(t) = v_t$ and $y(t) = z_t$ for all $t \in [0, L-1]$. Since (11.97) has full row rank, there exist $\bar{x}(0) \in \mathbb{R}^n$ and $\bar{u}(0), \bar{u}(1), \ldots, \bar{u}(i-1) \in \mathbb{R}^m$ such that

$$x(0) = \begin{bmatrix} A_{\text{true}}^i & A_{\text{true}}^{i-1} B_{\text{true}} & \cdots & A_{\text{true}} B_{\text{true}} & B_{\text{true}} \end{bmatrix} \begin{bmatrix} \bar{x}(0) \\ \bar{u}(0) \\ \vdots \\ \bar{u}(i-2) \\ \bar{u}(i-1) \end{bmatrix}. \tag{11.98}$$

Define $\bar{u}(t+i) := v_t$ for $t \in [0, L-1]$. Consider the restricted input-state-output trajectory $(\bar{u}(t), \bar{x}(t), \bar{y}(t))_{t=0}^{i+L-1}$ of (1.1) resulting from $\bar{x}(0)$ and $\bar{u}_{[0,i+L-1]}$. In view of (11.98), we obtain $\bar{x}(i) = x(0)$. Therefore, $\bar{y}_{[i,i+L-1]} = z$. We conclude that (v, z) is a restricted input-output trajectory of (1.1) on the interval $[i, i+L-1]$. This proves the claim, and therefore statement (c) of Theorem 1.2.

11.8.2 Proof of Proposition 11.2 (c)

By hypothesis of the proposition, we have that

$$T \geqslant L_+ + (L_+ + N_+ + 1)m + N_+. \tag{11.99}$$

Moreover, we recall that Proposition 11.2 (b) asserts that

$$\operatorname{rank} \begin{bmatrix} H_{L_++1}(u_{[0,T-1]}) \\ H_{L_++1}(y_{[0,T-1]}) \end{bmatrix} = (L_+ + 1)m + n_{\text{true}}. \tag{11.100}$$

We now want to show that the conditions (11.22) of Theorem 11.6 hold with $L_- = N_- = 0$. By definition of $\ell_{\min}$ and $n_{\min}$, (11.22a) and (11.22b) hold. Since $L_+ \geqslant L_+^{\text{a}}$ and $N_+ \geqslant n_{\min}$, (11.22c) follows from (11.99).

Let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{E}(n_{\min})$ and let $X_{[0,T]} \in \mathbb{R}^{n_{\min} \times (T+1)}$ be a state for $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Because $L_+ + 1 \geqslant \ell_{\min}$, we see from (11.100) that $\operatorname{rank} J_{L_++1}(X) = (L_+ + 1)m + n_{\text{true}}$. This implies that $n_{\text{true}} \leqslant n_{\min}$. Since the reverse inequality holds due to the definition of $n_{\min}$ in (11.19), we see that $n_{\min} = n_{\text{true}}$. Therefore, $J_{L_++1}(X)$ has full row rank. Since $L_+ \geqslant L_+^{\text{a}}$, we conclude from Lemma 11.15 (b) that $\operatorname{rank} J_{L_+^{\text{a}}+1}(X) = (L_+^{\text{a}}+1)m + n_{\min}$. Since $L_+^{\text{a}} + 1 \geqslant \ell_{\text{true}}$, we conclude that (11.22d) holds. In other words, by Theorem 11.6 the data $(u_{[0,T-1]}, y_{[0,T-1]})$ are informative for system identification within $\mathcal{S}_{[0,L_+],[0,N_+]} \cap \mathcal{M}$.

11.9 Notes and references

J.C. Willems' trilogy [186–188] is a deep and influential study on mathematical modelling of dynamical systems from time series. The second part [187] concerns the problem of obtaining a mathematical model for a linear system from a given (infinite) trajectory. It significantly influenced subspace identification methods [115], that compute a state sequence from finite-length data by adapting Willems' state construction from infinite to finite data. Two assumptions are crucial in [115]: the state-space dimension of the system is known; and a rank condition holds for a Hankel matrix constructed from the data. Although not formally proven at the time, it was believed that such rank condition is satisfied if the input data are sufficiently persistently exciting. This conjecture was formally proven in Willems et. al.'s *fundamental lemma* (see [190, Thm. 1] and Theorem 1.2) which allows the application of subspace identification even when only an upper bound on the state dimension is known.

As shown in this chapter (Proposition 11.2), the fundamental lemma gives a *sufficient* condition under which a linear time-invariant system can be uniquely identified from data. As shown by means of examples, however, this condition is not necessary. Motivated by this, we have investigated *necessary and sufficient* conditions on the input-output data for identifiability. Throughout, we have worked under the assumptions that the true system is minimal, and the lag and state-space dimension of the system are between given lower and upper bounds. The development of this chapter follows the paper [32].

Our approach is conceptually and methodologically close to the behavioral one (see [186–188, 190]). Instrumental to our results is the definition of a number of integer invariants computed directly from the data and associated with systems consistent with the data. Such integers are the finite data counterparts of those introduced in [186, Sect. 7] for *infinite* time series. Moreover, the fundamental lemma can be interpreted as a *special case* of our results.

An interesting consequence of Theorem 11.6 is in the context of online experiment design for system identification. Assuming that only an upper bound L_+ on the lag is known, [166, Thm. 3] gives a procedure to construct an input sequence $u_{[0,T-1]}$ with $T = (L_+ + 1)m + L_+ + n_{\text{true}}$ in such a way that the resulting data $(u_{[0,T-1]}, y_{[0,T-1]})$ are informative for system identification within $\mathcal{S}_{[0,L_+],[0,pL_+]} \cap \mathcal{M}$. What is striking is that such procedure does *not* require exact knowledge of n_{true}, even though the time horizon of the experiment depends on n_{true}. As discussed in Remark 11.8, in this case we have that $L_+^{\text{a}} = L_+$. If the data $(u_{[0,T-1]}, y_{[0,T-1]})$ are informative for system identification within $\mathcal{S}_{[0,L_+],[0,pL_+]} \cap \mathcal{M}$, we see from (11.22c) and (11.23b) that $T \geqslant (L_+ + 1)m + L_+ + n_{\text{true}}$. This proves that the experiment design procedure provided by [166, Thm. 3] generates the *minimal* number of samples required for system identification. We will study the problem of experiment design in more detail in the next chapter. In particular, a highlight of Chapter 12 is that even shorter experiments can be obtained if a bound N_+ on the state-space dimension is given, in addition to L_+.

12

Experiment design

In the previous chapter, we have provided necessary and sufficient conditions under which given input-output data $(u_{[0,T-1]}, y_{[0,T-1]})$ are informative for system identification. The main result of Chapter 11 is Theorem 11.6, which, roughly speaking, asserts that the data are informative if and only if a rank condition on a certain input-output Hankel matrix holds. In the current chapter, we follow up by answering the following question: how to design a sequence of inputs $u_{[0,T-1]}$ such that the input-output data are informative for system identification? Of course, a partial answer to this question has already been given by the fundamental lemma [190], see Proposition 11.2 in Chapter 11. The persistency of excitation condition of the fundamental lemma, however, imposes a conservative lower bound on the required number of data samples. Motivated by this, we are interested in designing the shortest possible experiments for system identification. To do so, we will see that it is important to design the inputs in an *online* manner, based on input-output data gathered in the past.

12.1 Informativity for system identification

We start by recapping the definition and characterization of informativity for system identification. As in Chapter 11, we consider the input-state-output system

$$x(t+1) = A_{\text{true}}x(t) + B_{\text{true}}u(t) \tag{12.1a}$$

$$y(t) = C_{\text{true}}x(t) + D_{\text{true}}u(t) \tag{12.1b}$$

where $A_{\text{true}} \in \mathbb{R}^{n_{\text{true}} \times n_{\text{true}}}$, $B_{\text{true}} \in \mathbb{R}^{n_{\text{true}} \times m}$, $C_{\text{true}} \in \mathbb{R}^{p \times n_{\text{true}}}$ and $D_{\text{true}} \in \mathbb{R}^{p \times m}$ are unknown. Also the state-space dimension $n_{\text{true}} \geqslant 0$ is unknown. However, $m, p \geqslant 1$ are known. We refer to (12.1) as the *true system*. We denote its lag by

$$\ell_{\text{true}} := \ell(C_{\text{true}}, A_{\text{true}}).$$

Throughout this chapter, we assume that the true system is minimal, i.e., both controllable and observable. In addition, we assume that upper bounds

$$L \geqslant \ell_{\text{true}} \quad \text{and} \quad N \geqslant n_{\text{true}} \tag{12.2}$$

are given on the true lag and state-space dimension, respectively.

Without making assumptions on the input, let $(u_{[0,t-1]}, y_{[0,t-1]})$ be data obtained from (12.1). Note that in this chapter, the data length is denoted by t, which will later on be variable in the online experiment design. By definition, there exists $X_{[0,t]} \in \mathbb{R}^{n_{\text{true}} \times (t+1)}$ such that

$$\begin{bmatrix} X_{[1,t]} \\ Y_{[0,t-1]} \end{bmatrix} = \begin{bmatrix} A_{\text{true}} & B_{\text{true}} \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix} \begin{bmatrix} X_{[0,t-1]} \\ U_{[0,t-1]} \end{bmatrix}. \tag{12.3}$$

Let $n \geqslant 0$. Recall from Chapter 11 that a system

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{S}(n) = \mathbb{R}^{(n+p)\times(n+m)}$$

is *consistent with the data* $(u_{[0,t-1]}, Y_{[0,t-1]})$ if there exists $X_{[0,t]} \in \mathbb{R}^{n\times(t+1)}$ such that

$$\begin{bmatrix} X_{[1,t]} \\ Y_{[0,t-1]} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_{[0,t-1]} \\ U_{[0,t-1]} \end{bmatrix}.$$

The set of all systems that are consistent with the data $(u_{[0,t-1]}, Y_{[0,t-1]})$ is denoted by $\mathcal{E}_t$ and is referred to as the *set of consistent systems*. The subsets of $\mathcal{E}_t$ consisting of systems with a given lag and state space dimension are respectively defined as

$$\mathcal{E}_t(\ell, n) := \mathcal{E}_t \cap \mathcal{S}(\ell, n) \quad \text{and} \quad \mathcal{E}_t(n) := \mathcal{E}_t \cap \mathcal{S}(n).$$

Here, we recall that the notation $\mathcal{S}(\ell, n)$ has been defined in (11.3) in Chapter 11.

12.1.1 Definition of informativity for system identification

The set $\mathcal{S}_{L,N}$ consists of all systems with lag at most L and state-space dimension at most N, i.e.,

$$\mathcal{S}_{L,N} := \bigcup_{\substack{\ell \in [0,\, L] \\ n \in [0,\, N]}} \mathcal{S}(\ell, n).$$

In view of the bounds (12.2) and the minimality of the true system, we have the following *prior knowledge*:

$$\begin{bmatrix} A_{\text{true}} & B_{\text{true}} \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix} \in \mathcal{M} \cap \mathcal{S}_{L,N},$$

where $\mathcal{M}$ is the set of minimal systems defined in (11.6). With this in mind, we recall the notion of informativity for system identification.

Definition 12.1. We say that the data $(u_{[0,t-1]}, Y_{[0,t-1]})$ *are informative for system identification*[1] if

(i) $\mathcal{E}_t \cap \mathcal{M} \cap \mathcal{S}_{L,N} = \mathcal{E}_t(n_{\text{true}}) \cap \mathcal{M} \cap \mathcal{S}_{L,N}$, and

(ii) $\mathcal{E}_t \cap \mathcal{M} \cap \mathcal{S}_{L,N}$ has the isomorphism property.

The first condition means that all data-consistent systems satisfying the prior knowledge have n_{true} states, while the second one asserts that any pair of such systems is isomorphic. Definition 12.1 thus captures the important property that there is precisely one equivalence class of state-space systems consistent with the input-output data. In what follows, we will recall conditions under which the data are informative for system identification. Before we can do so, we need to introduce two important integers, namely the shortest lag and minimum number of states.

12.1.2 The shortest lag and minimum number of states

Given the data $(u_{[0,t-1]}, y_{[0,t-1]})$, we define the following two integers that play a pivotal role in the characterization of informativity for system identification:

$$\begin{aligned}\ell_{\min,t} &:= \min\{\ell \geqslant 0 \mid \mathcal{E}_t(\ell, n) \neq \varnothing \text{ for some } n \geqslant 0\}\\ n_{\min,t} &:= \min\{n \geqslant 0 \mid \mathcal{E}_t(n) \neq \varnothing\}.\end{aligned}$$

As shown in [32] and Chapter 11, these integers admit a simple characterization in terms of the data. To explain this, let $k \in [1, t]$ and denote the Hankel matrix of k block rows constructed from the data $(u_{[0,t-1]}, Y_{[0,t-1]})$ by

$$H_{k,t} := \begin{bmatrix} H_k(u_{[0,t-1]}) \\ \hdashline H_k(y_{[0,t-1]}) \end{bmatrix} = \begin{bmatrix} u(0) & u(1) & \cdots & u(t-k) \\ \vdots & \vdots & & \vdots \\ u(k-1) & u(k) & \cdots & u(t-1) \\ \hdashline y(0) & y(1) & \cdots & y(t-k) \\ \vdots & \vdots & & \vdots \\ y(k-1) & y(k) & \cdots & y(t-1) \end{bmatrix}.$$

We also define

$$G_{k,t} := \begin{bmatrix} H_k(u_{[0,t-1]}) \\ H_{k-1}(Y_{[0,t-2]}) \end{bmatrix}.$$

[1]Since we only deal with the prior knowledge $\mathcal{M} \cap \mathcal{S}_{L,N}$, we will simply write 'informative for system identification' rather than 'informative for system identification within $\mathcal{M} \cap \mathcal{S}_{L,N}$'.

Note that $G_{k,t}$ may be obtained by removing the last row of outputs from $H_{k,t}$. Now, define

$$\delta_{k,t} := \begin{cases} p & \text{if } k = 0 \\ \operatorname{rank} H_{k,t} - \operatorname{rank} G_{k,t} & \text{if } k \in [1, t]. \end{cases}$$

Note that

$$p \geqslant \delta_{k,t} \geqslant 0 \text{ for all } k \in [0, t].$$

Throughout the chapter, we assume that $u_{[0,t-1]} \neq 0$, i.e., the inputs are not all equal to zero. From this blanket assumption, it follows that $\operatorname{rank} H_{t,t} = \operatorname{rank} G_{t,t} = 1$ and hence

$$\delta_{t,t} = 0. \tag{12.4}$$

Let $q_t \in [0, t-1]$ be the smallest integer such that $\delta_{q_t+1,t} = 0$. Note that q_t is well-defined due to (12.4). The shortest lag and minimum number of states can be computed in terms of $\delta_{k,t}$ and q_t, as recalled next (see [32, Thm. 8] and Theorem 11.5).

Proposition 12.2. *$\ell_{\min,t} = q_t$ and $n_{\min,t} = \sum_{i=1}^{\ell_{\min,t}} \delta_{i,t}$.*

An important consequence of Proposition 12.2 is that the integers $\ell_{\min,t}$ and $n_{\min,t}$ can be readily computed using the data.

12.1.3 Necessary and sufficient conditions for informativity

We are now in a position to recall the conditions for informativity for system identification. Before we do so, we remind the reader that Theorems 11.3.(b) and 11.5 show that if $\mathcal{E}_t(\ell, n) \neq \varnothing$ then

$$\ell \leqslant n - n_{\min,t} + \ell_{\min,t}. \tag{12.5}$$

This implies that $N - n_{\min,t} + \ell_{\min,t}$ is an upper bound for the lag of *any* consistent system with at most N states. This upper bound, which is determined by the data and N, is in some cases smaller than the given upper bound L. This means that we can replace L by the *actual* upper bound on the lag:

$$L_t^a := \min(L, N - n_{\min,t} + \ell_{\min,t}).$$

The following theorem is a reformulation of Theorem 11.6 and provides *necessary and sufficient* conditions for the data to be informative for system identification.

Theorem 12.3. *The data* $(u_{[0,t-1]}, y_{[0,t-1]})$ *are informative for system identification if and only if the following two conditions hold:*

$$t \geqslant L_t^a + (L_t^a + 1)m + n_{\min,t} \tag{12.6a}$$

$$\operatorname{rank} H_{L_t^a+1,t} = (L_t^a + 1)m + n_{\min,t}. \tag{12.6b}$$

Moreover, if the conditions in (12.6) *are satisfied, then*

$$\ell_{\text{true}} = \ell_{\min,t} \tag{12.7a}$$

$$n_{\text{true}} = n_{\min,t} \tag{12.7b}$$

$$\mathcal{E}_t \cap \mathcal{M} \cap \mathcal{S}_{L,N} = \mathcal{E}_t(n_{\min,t}). \tag{12.7c}$$

12.2 Formal problem statement

If the data $(u_{[0,t-1]}, y_{[0,t-1]})$ are informative for system identification, then $\ell_{\min,t} = \ell_{\text{true}}$ and $n_{\min,t} = n_{\text{true}}$ by Theorem 12.3. In this case, L_t^a is equal to

$$L^a := \min(L, N - n_{\text{true}} + \ell_{\text{true}}).$$

It follows from the lower bound (12.6a) that

$$t \geqslant T := L^a + (L^a + 1)m + n_{\text{true}}, \tag{12.8}$$

that is, any set of informative input-output data contains at least T samples.

The main question is now as follows: can we design a sequence of inputs $u_{[0,T-1]}$ of length *precisely* T such that the resulting input-output data

$$(u_{[0,T-1]}, y_{[0,T-1]})$$

are informative for system identification? We will focus on an *online* design of the inputs, in the sense that the choice of $u(t)$ is guided by the data $(u_{[0,t-1]}, y_{[0,t-1]})$ collected at previous time steps. We formalize the problem as follows.

Problem 12.4. Let T be as in (12.8). Consider the system (12.1) with initial state $x(0) = x_0 \in \mathbb{R}^n$. Let $y(t) \in \mathbb{R}^p$ denote the output of (12.1) at time t resulting from x_0 and the control inputs $u(0), u(1), \dots, u(t) \in \mathbb{R}^m$.

For each $t \in [0, T-1]$, given $(u_{[0,t-1]}, y_{[0,t-1]})$, design $u(t)$ such that, in the end, the resulting data $(u_{[0,T-1]}, y_{[0,T-1]})$ are informative for system identification.

We note that the initial state x_0 of the system (12.1) is arbitrary and not assumed to be given. Our goal is thus to design inputs that lead to an informative experiment *irrespective of* x_0.

In addition, we emphasize that it is not straightforward to see that Problem 12.4 has a solution. In fact, even though T is a lower bound on the number of data samples required for system identification, it is at this point unclear whether there *exists* an experiment of length exactly T. Also, even if such an experiment exists, it is far from obvious that there is a systematic way of *constructing* such an experiment without knowledge of the true system. An additional challenge is that the time T itself depends on the true lag and true state-space dimension, which are *not a priori known*.

Remarkably, as we show in this chapter, it turns out to be *always* possible to design an informative experiment of length precisely T, despite these challenges.

12.3 Online experiment design

In this section we present our main results, building up to the online experiment design method. We start with the following auxiliary lemma that asserts that the rank of the Hankel matrix $H_{k,t}$ can be increased at time $t+1$, assuming that certain conditions are met. To introduce the lemma, we will use the following terminology: a subset $\mathcal{A} \subseteq \mathbb{R}^n$ is called *affine* if it can be expressed as $\mathcal{A} = \{x\} + \mathcal{S}$ where $x \in \mathbb{R}^n$ and $\mathcal{S} \subseteq \mathbb{R}^n$ is a subspace. The dimension of $\mathcal{A}$ is defined as the dimension of $\mathcal{S}$.

Lemma 12.5. *Let $t \geqslant 2$ and $k \geqslant 2$. If*

$$\operatorname{rank} G_{k,t} < m + \operatorname{rank} H_{k-1,t} \tag{12.9}$$

then there exists an $(m-1)$-dimensional affine set $\mathcal{A}_t \subseteq \mathbb{R}^m$ such that

$$\operatorname{rank} H_{k,t+1} = \operatorname{rank} H_{k,t} + 1 \tag{12.10}$$

whenever $u(t) \notin \mathcal{A}_t$.

In what follows, we will use the shorthand notation $\mathbf{0}_m := \{0_{1,m}\}$.

Proof. Note that $\operatorname{lker} H_{k-1,t} \times \mathbf{0}_m \subseteq \operatorname{lker} G_{k,t}$. Therefore, it holds that

$$\dim \operatorname{lker} G_{k,t} \geqslant \dim \operatorname{lker} H_{k-1,t}.$$

It follows from the rank-nullity theorem that

$$kp + (k+1)m - \operatorname{rank} G_{k,t} \geqslant k(p+m) - \operatorname{rank} H_{k-1,t}.$$

As such, $\operatorname{rank} G_{k,t} \leqslant m + \operatorname{rank} H_{k-1,t}$. Moreover, note that $\operatorname{rank} G_{k,t} = m + \operatorname{rank} H_{k-1,t}$ if and only if $\operatorname{lker} H_{k-1,t} \times \mathbf{0}_m = \operatorname{lker} G_{k,t}$. Therefore, (12.9) implies that there exist $\eta_i \in \mathbb{R}^m$ and $\xi_j \in \mathbb{R}^p$ and with $i \in [1,k]$ and $j \in [1,k-1]$ such that $\eta_k \neq 0$ and

$$\begin{bmatrix}\eta_1^\top & \cdots & \eta_k^\top & \xi_1^\top & \cdots & \xi_{k-1}^\top\end{bmatrix} G_{k,t} = 0.$$

Now, define the set

$$\mathcal{A}_t := \{v \in \mathbb{R}^m \mid \eta_k^\top v + \sum_{i\in[1,k-1]} \eta_i^\top u(t-k+i) + \xi_i^\top y(t-k+i) = 0\}.$$

If $u(t) \notin \mathcal{A}_t$ then

$$\begin{bmatrix}\eta_1^\top & \cdots & \eta_k^\top & \xi_1^\top & \cdots & \xi_{k-1}^\top\end{bmatrix} G_{k,t+1} \neq 0.$$

Since $\operatorname{lker} G_{k,t+1} \subseteq \operatorname{lker} G_{k,t}$, we conclude from the latter inequality that

$$\dim \operatorname{lker} G_{k,t+1} < \dim \operatorname{lker} G_{k,t}.$$

Therefore, the last column of $G_{k,t+1}$ is not a linear combination of the columns of $G_{k,t}$. Thus, the last column of $H_{k,t+1}$ is also not a linear combination of the columns of $H_{k,t}$. We conclude that (12.10) holds, which proves the lemma. $\square$

As long as the inequality (12.9) holds, Lemma 12.5 may be successively applied several times to increase the rank of the Hankel matrix. In the next lemma, we show how to deduce from the data whether $k = L^a$, as soon as the condition (12.9) fails to hold. This lemma will be used as a stopping criterion for our online experiment design algorithm.

Lemma 12.6. *Let $t \geqslant 2$ and $k \geqslant 2$ and suppose that the data $(u_{[0,t-1]}, y_{[0,t-1]})$ are such that $H_{k,t}$ has full column rank. Let $\tau \geqslant 0$ and assume that $u_{[t,t+\tau-1]}$ satisfy*

(a) *for every $s \in [t, t+\tau-1]$, $\operatorname{rank} G_{k,s} < m + \operatorname{rank} H_{k-1,s}$ and $u(s) \notin \mathcal{A}_s$, where $\mathcal{A}_s$ is as in Lemma 12.5.*

(b) $\operatorname{rank} G_{k,t+\tau} = m + \operatorname{rank} H_{k-1,t+\tau}$.

Then the following statements hold:

(i) *If $k \geqslant \ell_{\text{true}} + 1$ then*

- $\operatorname{rank} H_{k,t+\tau} = n_{\text{true}} + km$,
- $t + \tau = n_{\text{true}} + km + k - 1$,
- $\ell_{\min,t+\tau} = \ell_{\text{true}}$, *and* $n_{\min,t+\tau} = n_{\text{true}}$.

(ii) *If $k = L^a_{t+\tau} + 1$ then*

- $k = L^a + 1$,
- $t + \tau = T$, *and*
- $(u_{[0,T-1]}, y_{[0,T-1]})$ *are informative for system identification.*

Proof. We first prove (i). Assume that $k \geqslant \ell_{\text{true}} + 1$. Hypothesis (a) and Lemma 12.5 imply that

$$\operatorname{rank} H_{k,t+\tau} = \operatorname{rank} H_{k,t} + \tau = t - k + \tau + 1. \tag{12.11}$$

Let $X_{[0,t+\tau-1]} \in \mathbb{R}^{n_{\text{true}} \times (t+\tau)}$ be a state compatible with the input-output data $(u_{[0,t+\tau-1]}, y_{[0,t+\tau-1]})$ and the true system. Since $(C_{\text{true}}, A_{\text{true}})$ is observable and $k \geqslant \ell_{\text{true}} + 1$, the observability matrix Ω_{k-1} of the true system (see Equation (11.2)) has rank n_{true}. This implies that the matrices

$$\Phi_{k-1} := \begin{bmatrix} 0 & I \\ \Omega_{k-1} & \Theta_{k-1} \end{bmatrix} \text{ and } \Psi_k := \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I_m \\ \Omega_{k-1} & \Theta_{k-1} & 0 \end{bmatrix} \tag{12.12}$$

have full column rank, where we recall that Θ_{k-1} is the Toeplitz matrix of Markov parameters of the true system, defined in (11.27). Therefore,

$$\operatorname{rank} H_{k-1,t+\tau} = \operatorname{rank}\left(\Phi_{k-1} \begin{bmatrix} X_{[0,t+\tau-k-1]} \\ H_{k-1}(u_{[0,t+\tau-1]}) \end{bmatrix}\right) = \operatorname{rank} \begin{bmatrix} X_{[0,t+\tau-k-1]} \\ H_{k-1}(u_{[0,t+\tau-1]}) \end{bmatrix}.$$

Moreover,

$$\begin{aligned} \operatorname{rank} G_{k,t+\tau} &= \operatorname{rank}\left(\Psi_k \begin{bmatrix} X_{[0,t+\tau-k]} \\ H_k(u_{[0,t+\tau-1]}) \end{bmatrix}\right) = \operatorname{rank} \begin{bmatrix} X_{[0,t+\tau-k]} \\ H_k(u_{[0,t+\tau-1]}) \end{bmatrix} && (12.13) \\ &= \operatorname{rank} H_{k,t+\tau}. && (12.14) \end{aligned}$$

By hypothesis (b), we thus have

$$\operatorname{rank} \begin{bmatrix} X_{[0,t+\tau-k]} \\ H_k(u_{[0,t+\tau-1]}) \end{bmatrix} = m + \operatorname{rank} \begin{bmatrix} X_{[0,t+\tau-k-1]} \\ H_{k-1}(u_{[0,t+\tau-1]}) \end{bmatrix}.$$

By Lemma 11.17 and the fact that $(A_{\text{true}}, B_{\text{true}})$ is controllable,

$$\operatorname{rank} \begin{bmatrix} X_{[0,t+\tau-k]} \\ H_k(u_{[0,t+\tau-1]}) \end{bmatrix} = n_{\text{true}} + km.$$

Therefore, by (12.14), we conclude that

$$\operatorname{rank} H_{k,t+\tau} = n_{\text{true}} + km, \tag{12.15}$$

proving the first item of (i). The second item of (i) now follows immediately from (12.11). Finally, to prove the third item, let

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{E}(\ell_{\min,t+\tau}, n_{\min,t+\tau}).$$

Let $X_{[0,t+\tau-1]} \in \mathbb{R}^{n_{\min,t+\tau} \times (t+\tau)}$ be a state compatible with the input-output data and the above data-consistent system. Obviously, $\operatorname{rank} H_{k,t+\tau} \leqslant n_{\min,t+\tau} + km$ and therefore $n_{\min,t+\tau} \geqslant n_{\text{true}}$ by (12.15). However, since also $n_{\min,t+\tau} \leqslant n_{\text{true}}$, we obtain $n_{\text{true}} = n_{\min,t+\tau}$. Finally, it follows from (12.5) that $\ell_{\min,t+\tau} \geqslant \ell_{\text{true}}$. Since obviously $\ell_{\min,t+\tau} \leqslant \ell_{\text{true}}$, we conclude that $\ell_{\min,t+\tau} = \ell_{\text{true}}$, proving the third item of (i).

Next, we will prove (ii). Assume that $k = L^a_{t+\tau} + 1$. We have that

$$N - n_{\min,t+\tau} + \ell_{\min,t+\tau} \geqslant n_{\text{true}} - n_{\min,t+\tau} + \ell_{\min,t+\tau} \geqslant \ell_{\text{true}},$$

where the last inequality follows from (12.5). Combining this with $L \geqslant \ell_{\text{true}}$, we obtain $k = L^a_{t+\tau} + 1 \geqslant \ell_{\text{true}} + 1$, by definition of $L^a_{t+\tau}$. Therefore, the three items listed under (i) hold. From the fact that $\ell_{\min,t+\tau} = \ell_{\text{true}}$ and $n_{\min,t+\tau} = n_{\text{true}}$, it follows that $L^a_{t+\tau} = L^a$, and therefore $k = L^a + 1$. This proves the first item of (ii). Moreover, the second item of (i) implies that $t + \tau = L^a + (L^a + 1)m + n_{\text{true}}$, which shows that $t + \tau = T$, proving the second item of (ii). Finally, from the first and third items of (i), we see that $\operatorname{rank} H_{L^a_T+1,T} = (L^a_T + 1)m + n_{\min,T}$. Therefore, it follows from Theorem 12.3 that the data $(u_{[0,T-1]}, y_{[0,T-1]})$ are informative for system identification. Hence, also the third item of (ii) holds. This proves the lemma. ☐

The core idea of our approach is to adapt the depth k of the Hankel matrix during the operation of the experiment design procedure. For a *fixed* depth k, Lemma 12.5 will be used for $s \in [t, t + \tau - 1]$ until the rank condition $\operatorname{rank} G_{k,t+\tau} = m + \operatorname{rank} H_{k-1,t+\tau}$ holds. Then, following Lemma 12.6, we check whether $k = L^a_{t+\tau} + 1$. If $k = L^a_{t+\tau} + 1$ then we are done because the data $(u_{[0,t+\tau-1]}, y_{[0,t+\tau-1]})$ are informative for system identification. Otherwise, if $k \neq L^a_{t+\tau} + 1$ we increase the depth of the Hankel matrix to $k + 1$ and repeat the process. This leads to the following algorithm.

procedure ONLINEEXPERIMENT(L, N)
  choose inputs $u_{[0,m-1]}$ such that $U_{[0,m-1]}$ is nonsingular
  measure outputs $y_{[0,m-1]}$
  $t \leftarrow m, k \leftarrow 1$
  while $k \neq L^a_t + 1$ **do** ▷ stopping criterion
    $k \leftarrow k + 1$
    if $k = t + 1$ **then**
      choose $u(t)$ arbitrarily

9: **measure** output $y(t)$
10: $t \leftarrow t+1$
11: **end if**
12: **while** rank $G_{k,t} < m + \operatorname{rank} H_{k-1,t}$ **do**
13: **choose** $u(t) \notin \mathcal{A}_t$
14: **measure** output $y(t)$ ▷ rank $H_{k,t+1} = \operatorname{rank} H_{k,t} + 1$
15: $t \leftarrow t+1$
16: **end while**
17: **end while**
18: **return** $(U_{[0,t-1]}, Y_{[0,t-1]})$ ▷ the data are informative
19: **end procedure**

The following theorem asserts that ONLINEEXPERIMENT(L, N) leads to informative data sets with the least possible number of samples. This is the main result of the chapter.

Theorem 12.7. *The procedure* ONLINEEXPERIMENT(L, N) *returns input-output data* $(U_{[0,t-1]}, Y_{[0,t-1]})$ *that are informative for system identification. Moreover,* $t = T$*, where* T *is defined in* (12.8).

Before we prove Theorem 12.7, we state the following auxiliary lemma.

Lemma 12.8. *Let* $k \geqslant 1$ *and* $t \geqslant k+1$*. If the data* $(u_{[0,t-1]}, y_{[0,t-1]})$ *are such that* $H_{k,t}$ *has full column rank then also* $H_{k+1,t}$ *has full column rank.*

Proof. Since $H_{k,t}$ has full column rank, the submatrix

$$\begin{bmatrix} u(1) & \cdots & u(t-k) \\ \vdots & & \vdots \\ u(k) & \cdots & u(t-1) \\ y(1) & \cdots & y(t-k) \\ \vdots & & \vdots \\ y(k) & \cdots & y(t-1) \end{bmatrix}, \tag{12.16}$$

obtained from removing the first column of $H_{k,t}$, has full column rank as well. Since (12.16) is also the submatrix of $H_{k+1,t}$ obtained by removing the row blocks $U_{[0,t-k-1]}$ and $Y_{[0,t-k-1]}$, we conclude that $H_{k+1,t}$ has full column rank. This proves the lemma. □

Proof of Theorem 12.7. The proof consists of the following three steps. First, we prove that the Hankel matrix $H_{k,t}$ always has full column rank at the start of the while loop in Line 12. Secondly, we prove that the procedure terminates within a finite number of steps. Finally, we show that the latter number is precisely equal to T, and the data $(u_{[0,T-1]}, y_{[0,T-1]})$ are informative for system identification.

We begin with the first step. Consider the while loop in Lines 5–17. Let k be the depth of the Hankel matrix afer line 6. Moreover, let t_k be the time instant at the start of the while loop in Line 12. We claim that H_{k,t_k} has full column rank.

We first prove this claim for the first iteration of the while loop, i.e., consider $k = 2$. If $m \geqslant 2$ then the if statement in Lines 7–11 is ignored, and $t_2 = m$. Note that the Hankel matrix $H_{1,m}$ has full column rank by the choice of the inputs $u_{[0,m-1]}$ in Line 2. It is then clear that H_{2,t_2} has full column rank by Lemma 12.8. On the other hand, if $m = 1$, then $t_2 = m + 1$. In this case, the Hankel matrix $H_{2,m+1}$ has one column, which is nonzero due to line 2. Therefore, also in this case H_{2,t_2} has full column rank.

Now consider any $k \geqslant 2$. Assume that H_{k,t_k} has full column rank. Our goal is to show that $H_{k+1,t_{k+1}}$ has full column rank as well.

Since H_{k,t_k} has full column rank by hypothesis, we can apply Lemma 12.5 to the while loop in Lines 12–16. In particular, this while loop is applied for a finite number of iterations, say $\tau \in \mathbb{Z}_+$, which yields the Hankel matrix $H_{k,t_k+\tau}$. By repeated application of Lemma 12.5, $H_{k,t_k+\tau}$ has full column rank. This means, in particular, that $t_k + \tau \geqslant k$.

Now, we turn our attention to the depth $k + 1$. If $t_k + \tau = k$ then the if statement in Lines 7–11 generates an arbitrary input $u(t_k+\tau)$ and corresponding output $y(t_k + \tau)$. In this case, $t_{k+1} = t_k + \tau + 1$ and the resulting Hankel matrix $H_{k+1,t_{k+1}}$ is a column vector of rank one by Line 2. In the other case, if $t_k + \tau \geqslant k + 1$ then $t_{k+1} = t_k + \tau$ and it follows that $H_{k+1,t_{k+1}}$ has full column rank by Lemma 12.8.

Secondly, we prove that the procedure terminates in a finite number of steps. Now, let $t_k \in \mathbb{N}$ be the time instant, corresponding to the depth k, at which the stopping criterion in Line 5 is checked. We want to prove the existence of a depth $k \geqslant 1$ such that $k = L^a_{t_k} + 1$. Clearly, since $L^a_{t_k} \geqslant \ell_{\text{true}}$, this cannot happen if $k \leqslant \ell_{\text{true}}$. For any $k \geqslant \ell_{\text{true}} + 1$ we have that $\ell_{\min,t_k} = \ell_{\text{true}}$ and $n_{\min,t_k} = n_{\text{true}}$ by Lemma 12.6(i), meaning that $L^a_{t_k} = L^a$. Since the depth k is increased by one in every iteration of the while loop in Lines 5–17, this implies that there exists $k \geqslant \ell_{\text{true}} + 1$ such that $k = L^a_{t_k} + 1$.

Finally, we prove the last step. Let k be such that $k = L^a_{t_k} + 1$. It follows from Lemma 12.6(ii) that t_k is precisely equal to T, and the data $(u_{[0,T-1]}, y_{[0,T-1]})$ are informative for system identification. This proves the theorem. □

Remark 12.9. Without going into details, we mention that Theorem 12.7 shows that 'randomly' chosen inputs $u_{[0,T-1]}$ lead to informative experiments of length precisely T with high probability. In fact, the only imposed constraints on the inputs are that $U_{[0,m-1]}$ is nonsingular (Line 2 of the algorithm), and that $u(t) \in \mathbb{R}^m$ is not a member of an $(m-1)$-dimensional affine set (Line 13). Regardless of how the inputs are chosen to satisfy these constraints, however, a crucial aspect of OnlineExperiment(L, N) is its stopping criterion. Indeed, we emphasize that T is not a priori known but has to be deduced from data.

12.4 Illustrative example

In this section we illustrate OnlineExperiment(L, N) by means of an example. Consider the true system

$$\begin{bmatrix} A_{\text{true}} & B_{\text{true}} \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix} = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right],$$

so that $n_{\text{true}} = 3$, $m = 2$, $p = 2$, and $\ell_{\text{true}} = 2$. Let $x_0 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top$. Moreover, let $L = 4$ and $N = 4$ be the available upper bounds.

12.4.1 Online experiment

We now apply OnlineExperiment(L, N). We start with $k = 1$. We choose the first two inputs such that $U_{[0,1]} = I$, which is obviously nonsingular, and measure

$$Y_{[0,1]} = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}.$$

Using these data and Proposition 12.2, we compute

$$\ell_{\min,2} = 0, \;\; n_{\min,2} = 0, \;\; \text{and} \;\; L_2^a = 4.$$

Since $k \neq L_2^a + 1$, we increase the depth to $k = 2$. The inputs are now designed according to the while loop in Line 12 as:

$$U_{[2,7]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

resulting in the measured outputs

$$Y_{[2,7]} = \begin{bmatrix} 1 & 3 & 3 & 2 & 2 & 2 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It can be checked that we have increased the rank of the depth-2 Hankel matrix from rank $H_{2,3} = 2$ to rank $H_{2,8} = 7$. Based on the data thusfar, we use Proposition 12.2 to compute:

$$\ell_{\min,8} = 2, \;\; n_{\min,8} = 3, \;\; \text{and} \;\; L_8^a = 3.$$

Since $k \neq L^a_8 + 1$, we set $k = 3$. Following the while loop in Line 12, we construct

$$U_{[8,10]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

which yields

$$Y_{[8,10]} = \begin{bmatrix} 2 & 3 & 3 \\ 1 & 1 & 0 \end{bmatrix}.$$

By doing so, we have increased the rank of the depth-3 Hankel matrix from $\operatorname{rank} H_{3,9} = 7$ to $\operatorname{rank} H_{3,11} = 9$. Again, we compute:

$$\ell_{\min,11} = 2, \quad n_{\min,11} = 3, \quad \text{and} \quad L^a_{11} = 3.$$

Since $k \neq L^a_{11} + 1$ we set $k = 4$. This time, we apply the while loop in Line 12 to obtain the data

$$U_{[11,13]} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \text{ and } Y_{[11,13]} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The rank of the depth-4 Hankel matrix has increased from $\operatorname{rank} H_{4,12} = 9$ to $\operatorname{rank} H_{4,14} = 11$. Finally, we compute

$$\ell_{\min,14} = 2, \quad n_{\min,14} = 3, \quad \text{and} \quad L^a_{14} = 3.$$

Since $k = L^a_{14} + 1$, the procedure terminates. We conclude that $T = 14$ and the data $(u_{[0,13]}, y_{[0,13]})$ are informative for system identification. For this example, we note that the required number of samples T is less than the experiment design approach in [166] that works with the *fixed* depth $L = 4$ Hankel matrix. However, this is not always the case. For example, if we study the same example but with the given upper bounds $L = 3$ and $N = 6$, we can use OnlineExperiment(L, N) to generate the same informative data, with the only difference that we now have $L^a_2 = 3$. In this case, the number of $T = 14$ data samples is the same as in [166].

12.4.2 PE of order $L^a + 1$ is not sufficient for informativity

According to Theorem 12.3, an obvious necessary condition for informativity is that the inputs are *persistently exciting of order* $L^a + 1$. This condition, however, is not sufficient as demonstrated next. We use the same example as above, but just change $u(12)$ from $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, i.e., we choose the inputs as

$$U_{[0,13]} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The corresponding outputs are then given by

$$Y_{[0,13]} = \begin{bmatrix} 2 & 2 & 1 & 3 & 3 & 2 & 2 & 2 & 2 & 3 & 3 & 2 & 2 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case, $\operatorname{rank} H_4(u_{[0,13]}) = 8$ so $u_{[0,13]}$ is persistently exciting of order $L^a + 1$. However $\operatorname{rank} H_{4,14} = 10 \neq 11$ so the conditions of Theorem 12.3 are not satisfied.

12.5 Notes and references

Experiment design is a classical problem that has been mostly studied in the parametric identification literature. An established idea is to optimize a measure of the expected accuracy of the parameter estimates subject to input power constraints [62,64,77]. This problem is usually tackled in the frequency domain and convex formulations have been provided in [82]. The dual problem of finding the 'least costly' input achieving a fixed level of parameter accuracy has also been studied [25], in a closed-loop setting.

The results in this chapter are based on the paper [33]. They align well with the subspace identification literature [165,178], where rank conditions on Hankel matrices of input-output data play a vital role. In this context, a state-of-the-art experiment design result is Willems et al.'s fundamental lemma [190], see Theorem 1.2 and Proposition 11.2.

The online experiment design method proposed in this chapter largely improves the (offline) persistency of excitation condition of the fundamental lemma. Indeed, recall that, by definition, the input $u_{[0,t-1]}$ can only be *persistently exciting* of order $N + L + 1$ if $t \geqslant N + L + m(N + L + 1)$. In general, this lower bound on the number of data samples is much larger than the time horizon T in (12.8). For example, if $m = 80$, $p = 10$, $\ell_{\text{true}} = 20$, $n_{\text{true}} = 100$, $L = 100$ and $N = 150$, the online experiment design method requires $T = 5850$ samples whereas persistency of excitation requires $t \geqslant 20330$ samples.

The proposed approach also improves the online experiment design of [166]. In fact, in the latter paper a method was given to guarantee that the Hankel matrix H_{L+1} of *fixed depth* $L + 1$ has rank $(L + 1)m + n_{\text{true}}$. This was done in the least possible number of time steps, $t = L + (L + 1)m + n_{\text{true}}$. However, by Theorem 12.3, the condition $\operatorname{rank} H_{L+1} = (L + 1)m + n_{\text{true}}$ is sufficient for informativity for system identification, but in general not necessary. In particular, if $L^a < L$ then the experiment design method of this chapter leads to a shorter experiment for system identification than the one provided in [166]. If $L^a = L$, then the number of samples coincides with [166].

Part IV

Data-driven Model Reduction

13

Data-driven model reduction by balanced truncation

For a given mathematical model of a dynamical system, the *model reduction problem* is to approximate this model by a lower-order, less complex one while preserving its essential properties. Conventional model reduction techniques derive these reduced-order models from the original model through operations such as, for example, projections. In contrast to this, in this chapter we develop methods that construct reduced-order models *directly from data obtained from the system*, without using a mathematical model of the system. Specifically, in this chapter we focus on *generalized Lyapunov balancing*, and investigate conditions under which input-state data are informative for this type of balancing.

13.1 Model reduction

In this section, we will briefly review some basic notions from the theory of model reduction. We refer the reader to [7] for a comprehensive overview of various existing model reduction methods.

Consider the linear input-state-output system

$$x(t+1) = Ax(t) + Bu(t) \tag{13.1a}$$

$$y(t) = Cx(t) + Du(t) \tag{13.1b}$$

with state-space dimension n, input dimension m and output dimension p. Let $G(s) = C(sI - A)^{-1}B + D$ denote the transfer matrix of this system. When n is very large, it is often preferable to replace the system (13.1) by a lower order one, say with state space dimension r, with r preferably much smaller than n, of the form

$$\widehat{x}(t+1) = A_{\text{red}}\widehat{x}(t) + B_{\text{red}}u(t) \tag{13.2a}$$

$$y(t) = C_{\text{red}}\widehat{x}(t) + Du(t). \tag{13.2b}$$

Let $G_{\text{red}}(s) = C_{\text{red}}(sI - A_{\text{red}})^{-1}B_{\text{red}} + D$ denote the transfer matrix of this system. The reduced-order system (13.2) should preserve certain properties and approximately retain the input-output behavior of the original system (13.1).

Among a multitude of existing model reduction schemes, in this chapter we will deal with so-called Petrov-Galerkin projection and generalized Lyapunov balancing.

13.1.1 Petrov-Galerkin projection

Given matrices $W, V \in \mathbb{R}^{n\times r}$ satisfying $W^\top V = I_r$, Petrov-Galerkin projection leads to a reduced-order model (13.2) with

$$A_{\mathrm{red}} = W^\top AV, \quad B_{\mathrm{red}} = W^\top B, \quad \text{and} \quad C_{\mathrm{red}} = CV. \tag{13.3}$$

Many reduction techniques, including Gramian- and Krylov-based methods can be regarded as Petrov-Galerkin projections with appropriate choices of W and V, see e.g. [7] for more details. A particularly important special case of Petrov-Galerkin projection is the so-called method of Lyapunov balancing that we will quickly recap next.

13.1.2 Lyapunov balancing

Suppose that A is stable[1] and the system (13.1) is minimal, i.e., controllable and observable. Then, the Lyapunov equations

$$P - APA^\top = BB^\top \tag{13.4}$$

$$Q - A^\top QA = C^\top C \tag{13.5}$$

admit unique positive definite solutions P and Q, known as the controllability Gramian and the observability Gramian, respectively. It is known (see e.g. [7, Lemma 4.29]) that the minimal input energy required to reach a state $\bar{x}$ in the state space $\mathbb{R}^n$ from the origin is given by the quantity

$$\bar{x}^\top P^{-1}\bar{x}.$$

In addition, the maximal observation energy produced by an initial state $\bar{x}$ is given by

$$\bar{x}^\top Q\bar{x}.$$

As such, these Gramians provide measures for reachability and observability of a given state in terms of energy. Indeed, the states that lie in the eigenspace corresponding to the smallest (largest) eigenvalue of P (Q) can be interpreted as those that are hardest to reach (observe). The main idea behind Lyapunov balancing is to perform a state transformation on the original system in such a

[1] Recall that a square matrix is called stable if all its eigenvalues λ satisfiy $|\lambda| < 1$.

way that states of the transformed system are equally difficult to reach and to observe. This can be achieved in the following way. First, note that

$$PQ = P^{\frac{1}{2}}(P^{\frac{1}{2}}QP^{\frac{1}{2}})P^{-\frac{1}{2}}.$$

Therefore, PQ is similar to the positive definite matrix $P^{\frac{1}{2}}QP^{\frac{1}{2}}$ and hence only has positive eigenvalues. The square roots of these eigenvalues are called the *Hankel singular values* of the system (13.1). We order these square roots as $\sigma_1 > \sigma_2 > \cdots > \sigma_\kappa > 0$. For $i \in [1, \kappa]$, the algebraic multiplicity of σ_i^2 as an eigenvalue of PQ is denoted by m_i. Clearly, $n = \sum_{i=1}^{\kappa} m_i$. Let[2]

$$H_{\text{sv}} := \text{Bdiag}(\sigma_1 I_{m_1}, \sigma_2 I_{m_2}, \ldots, \sigma_\kappa I_{m_\kappa}). \tag{13.6}$$

Let U be an orthogonal matrix such that

$$P^{\frac{1}{2}}QP^{\frac{1}{2}} = UH_{\text{sv}}^2 U^\top$$

and let

$$S = H_{\text{sv}}^{\frac{1}{2}} U^\top P^{-\frac{1}{2}}.$$

By using S as a state-space transformation, we obtain the *balanced* system $(A_{\text{bal}}, B_{\text{bal}}, C_{\text{bal}}, D)$ where

$$A_{\text{bal}} = SAS^{-1}, \quad B_{\text{bal}} = SB, \quad C_{\text{bal}} = CS^{-1}. \tag{13.7}$$

This system is balanced in the sense that its reachability and observability Gramians are diagonal and equal:

$$P_{\text{bal}} = SPS^\top = H_{\text{sv}} = S^{-\top}QS^{-1} = Q_{\text{bal}}.$$

A reduced-order model can be obtained from the original system (13.1) by applying the Petrov-Galerkin projection with

$$W = S^\top \begin{bmatrix} I_r \\ 0 \end{bmatrix} \quad \text{and} \quad V = S^{-1} \begin{bmatrix} I_r \\ 0 \end{bmatrix} \tag{13.8}$$

which simply truncates the last $n - r$ states of the balanced system. Since the Gramians P_{bal} and Q_{bal} are equal and diagonal, this truncation corresponds to eliminating states that are harder to reach and observe at the same time. The method of obtaining a reduced-order model based on the balanced system is known as Lyapunov balanced truncation. Reduced-order systems obtained by balanced truncation inherit certain properties of the original system. In addition, one can quantify to what extent they retain the input-output behavior of the original system in terms of the neglected Hankel singular values as stated next.

[2]Given matrices $M_1, M_2, \ldots, M_k$, the matrix $\text{Bdiag}(M_1, M_2, \ldots, M_k)$ denotes the block diagonal matrix with $M_1, M_2, \ldots, M_k$ as diagonal blocks.

Proposition 13.1 ([7, Thm. 7.10]). *Consider the system* (13.1)*. Assume that A is stable, (A,B) is controllable, and (C,A) is observable. Suppose that the reduced-order system* (13.2) *is obtained via Lyapunov balanced truncation. Then, the following statements hold:*

(a) *A_{red} is Schur.*

(b) *$(A_{\mathrm{red}}, B_{\mathrm{red}})$ is controllable, $(C_{\mathrm{red}}, A_{\mathrm{red}})$ is observable, and*

$$\|G - G_{\mathrm{red}}\|_{\mathcal{H}_\infty} \leqslant 2 \sum_{i=k+1}^{\kappa} \sigma_i$$

provided that $r = \sum_{i=1}^{k} m_i$ with $k < \kappa$.

Generalized Lyapunov balancing

In this chapter, we will be mainly interested in a variation of the Lyapunov balancing method described above, namely *generalized Lyapunov balancing* (GLB). Consider, instead of the Lyapunov equations (13.4), the following Lyapunov *inequalities*:

$$\hat{P} - A\hat{P}A^\top > BB^\top \tag{13.9}$$

$$\hat{Q} - A^\top\hat{Q}A > C^\top C. \tag{13.10}$$

These inequalities admit positive definite solutions $\hat{P}$ and $\hat{Q}$ provided that A is stable. Any such $\hat{P}$ ($\hat{Q}$) is called *a generalized controllability* (*observability*) *Gramian*. Any generalized Gramian is an upper bound on the ordinary Gramian, i.e., $\hat{P} \geqslant P$ and $\hat{Q} \geqslant Q$. For any choice $\hat{P}$ and $\hat{Q}$ of generalized Gramians, similar arguments as in the case of ordinary Lyapunov balancing yield a diagonal matrix $\hat{H}_{\mathrm{sv}}$

$$\hat{H}_{\mathrm{sv}} := \mathrm{Bdiag}(\hat{\sigma}_1 I_{\hat{m}_1}, \hat{\sigma}_2 I_{\hat{m}_2}, \ldots, \hat{\sigma}_{\hat{\kappa}} I_{\hat{m}_{\hat{\kappa}}}) \tag{13.11}$$

where $\hat{\sigma}_1 > \hat{\sigma}_2 > \cdots > \hat{\sigma}_{\hat{\kappa}} > 0$ are the so-called *generalized Hankel singular values* corresponding to the choice of Gramians $\hat{P}$ and $\hat{Q}$. Here, $\hat{m}_i$ is the algebraic multiplicity of $\hat{\sigma}_i^2$ as an eigenvalue of $\hat{P}\hat{Q}$ for $i \in [1, \hat{\kappa}]$. Again, $n = \sum_{i=1}^{\hat{\kappa}} \hat{m}_i$. Similar to ordinary Lyapunov balancing, there exists a nonsingular matrix $\hat{S}$ such that

$$\hat{S}\hat{P}\hat{S}^\top = \hat{H}_{\mathrm{sv}} = \hat{S}^{-\top}\hat{Q}\hat{S}^{-1}.$$

Then, a reduced-order model can be obtained from the original system by using the Petrov-Galerkin projection with

$$W = \hat{S}^\top \begin{bmatrix} I_r \\ 0 \end{bmatrix} \quad \text{and} \quad V = \hat{S}^{-1} \begin{bmatrix} I_r \\ 0 \end{bmatrix}. \tag{13.12}$$

This method is known as the *generalized Lyapunov balanced truncation*, for which the counterpart of Proposition 13.1 can be stated as follows.

Proposition 13.2. *Consider the system (13.1). Assume that A is stable, (A, B) is controllable, and (C, A) is observable. Suppose that the reduced-order system (13.2) is obtained via generalized Lyapunov balanced truncation. Then, the following statements hold:*

(a) *A_{red} is stable.*

(b) *$(A_{\text{red}}, B_{\text{red}})$ is controllable, $(C_{\text{red}}, A_{\text{red}})$ is observable, and*

$$\|G - G_{\text{red}}\|_{\mathcal{H}_\infty} \leqslant 2 \sum_{i=k+1}^{\hat{\kappa}} \hat{\sigma}_i$$

provided that $r = \sum_{i=1}^{k} \hat{m}_i$ with $k < \hat{\kappa}$.

A proof of this proposition follows *mutatis mutandis* from [7, Thm. 7.10] or [48, Prop. 4.19].

13.2 Data-driven model reduction

Consider the linear input-state-output system

$$x(t+1) = A_{\text{true}}x(t) + B_{\text{true}}u(t) + z_x(t) \tag{13.13a}$$
$$y(t) = C_{\text{true}}x(t) + D_{\text{true}}u(t) + z_y(t) \tag{13.13b}$$

with state-space dimension n, input dimension m, and output dimension p. In (13.13), z_x and z_y are noise terms. In the rest of this chapter, we assume that the system matrices $(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})$ and the noise (z_x, z_y) are *unknown*. What is known instead are a finite number of input-state-output measurements harvested from the true system (13.13):

$$\begin{gathered} u(0), u(1), \ldots, u(T-1) \\ x(0), x(1), \ldots, x(T) \\ y(0), y(1), \ldots, y(T-1). \end{gathered}$$

As usual, we collect these data in the matrices

$$X := X_{[0,T]}, \quad X_- := X_{[0,T-1]}, \quad X_+ := X_{[1,T]},$$
$$U_- := U_{[0,T-1]}, \quad \text{and} \quad Y_- := Y_{[0,T-1]}.$$

The noise sequences z_x and z_y are unknown, so

$$z_x(0), z_x(1), \dots, z_x(T-1)$$
$$z_y(0), z_y(1), \dots, z_y(T-1)$$

are not measured. However, the noise matrix

$$Z_- := \begin{bmatrix} z_x(0) & z_x(1) & \cdots & z_x(T-1) \\ z_y(0) & z_y(1) & \cdots & z_y(T-1) \end{bmatrix}$$

is assumed to satisfy

$$Z_-^\top \in \mathcal{Z}_T(\Phi) \tag{13.14}$$

where $\mathcal{Z}_T(\Phi)$ is defined as in (A.3) and where $\Phi \in \mathbb{S}^{n+p+T}$ is a known, given partitioned matrix

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \tag{13.15}$$

with $\Phi_{11} \in \mathbb{S}^{n+p}$, $\Phi_{12} \in \mathbb{R}^{(n+p)\times T}$, and $\Phi_{22} \in \mathbb{S}^T$. We assume that $\Phi \in \mathbf{\Pi}_{n+p,T}$ as defined in (A.11), i.e. $\Phi_{22} \leqslant 0, \Phi \,|\, \Phi_{22} \geqslant 0$ and $\ker \Phi_{22} \subseteq \ker \Phi_{12}$.

Now, the set of all systems that are consistent with the data $\mathcal{D} = (U_-, X, Y_-)$ is equal to

$$\Sigma_{\mathcal{D}} := \left\{ (A, B, C, D) \mid \left(\begin{bmatrix} X_+ \\ Y_- \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \right)^\top \in \mathcal{Z}_T(\Phi) \right\}.$$

As the data collected from the true system, we have

$$(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}}) \in \Sigma_{\mathcal{D}}. \tag{13.16}$$

It is clear from the definition of $\Sigma_{\mathcal{D}}$ that $(A, B, C, D) \in \Sigma_{\mathcal{D}}$ if and only if the following quadratic matrix inequality is satisfied

$$\begin{bmatrix} I & 0 \\ 0 & I \\ A^\top & C^\top \\ B^\top & D^\top \end{bmatrix}^\top N \begin{bmatrix} I & 0 \\ 0 & I \\ A^\top & C^\top \\ B^\top & D^\top \end{bmatrix} \geqslant 0 \tag{13.17}$$

where

$$N := \begin{bmatrix} I & 0 & X_+ \\ 0 & I & Y_- \\ 0 & 0 & -X_- \\ 0 & 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & 0 & X_+ \\ 0 & I & Y_- \\ 0 & 0 & -X_- \\ 0 & 0 & -U_- \end{bmatrix}^\top. \tag{13.18}$$

The QMI (13.17) can be written more compactly as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^\top = \begin{bmatrix} A^\top & C^\top \\ B^\top & D^\top \end{bmatrix} \in \mathcal{Z}_{n+m}(N). \tag{13.19}$$

Later on, we will often use the following partitioning of N

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

where $N_{11} \in \mathbb{S}^{n+p}$, $N_{12} \in \mathbb{R}^{(n+p)\times(n+m)}$, $N_{21} = N_{12}^\top$ and $N_{22} \in \mathbb{S}^{n+m}$. Note that

$$N_{22} = \begin{bmatrix} X_- \\ U_- \end{bmatrix} \Phi_{22} \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top \quad \text{and} \quad N_{12} = -\Phi_{12} \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top - \begin{bmatrix} X_+ \\ Y_- \end{bmatrix} \Phi_{22} \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top.$$

Since $\Phi \in \mathbf{\Pi}_{n+p,T}$, we have $\Phi_{22} \leqslant 0$, $\ker \Phi_{22} \subseteq \ker \Phi_{12}$, and $\Phi \mid \Phi_{22} \geqslant 0$. Therefore, we see that $N_{22} \leqslant 0$ and $\ker N_{22} \subseteq \ker N_{12}$. Due to (13.16), $\mathcal{Z}_{n+m}(N)$ is nonempty. It therefore follows from (A.10) that $N \mid N_{22} \geqslant 0$. Consequently, we see that

$$N \in \mathbf{\Pi}_{n+p,n+m}. \tag{13.20}$$

In the remainder of this chapter, the following assumption will be in force.

Assumption 13.3. The matrix N is nonsingular.

This assumption, together with (13.20), implies that both N_{22} and $N \mid N_{22}$ are nonsingular. As a consequence we have $N_{22} < 0$ and $N \mid N_{22} > 0$. As such, the set of data-consistent systems $\Sigma_\mathcal{D}$ is bounded and has nonempty interior due to Theorem A.5.

13.3 Data reduction via Petrov-Galerkin projection

In this section, we deal with Petrov-Galerkin projections of data-consistent systems. Consider the set of all reduced-order systems obtained from the set of data-consistent systems via Petrov-Galerkin projection with the matrices $V, W \in \mathbb{R}^{n\times r}$:

$$\Sigma_\mathcal{D}^{\text{red}}(W,V) := \left\{ (W^\top AV, W^\top B, CV, D) \mid (A,B,C,D) \in \Sigma_\mathcal{D} \right\}. \tag{13.21}$$

This set itself is a QMI induced set.

Theorem 13.4. *It holds that*

$$\Sigma_\mathcal{D}^{\text{red}}(W,V) = \left\{ (A_{\text{red}}, B_{\text{red}}, C_{\text{red}}, D_{\text{red}}) \mid \begin{bmatrix} A_{\text{red}} & B_{\text{red}} \\ C_{\text{red}} & D_{\text{red}} \end{bmatrix}^\top \in \mathcal{Z}_{r+m}(N_{\text{red}}(W,V)) \right\}$$

where

$$N_{\rm red}(W,V) := \left[\begin{array}{c:c} \hat{W}^\top \Delta \hat{W} & \hat{W}^\top N_{12} N_{22}^{-1} \hat{V} \Gamma \\ \hdashline \Gamma \hat{V}^\top N_{22}^{-1} N_{12}^\top \hat{W} & \Gamma \end{array}\right], \tag{13.22}$$

$$\Delta = N|N_{22} + N_{12}N_{22}^{-1}\hat{V}\Gamma\hat{V}^\top N_{22}^{-1}N_{12}^\top, \qquad \Gamma = (\hat{V}^\top N_{22}^{-1}\hat{V})^{-1},$$
$$\hat{W} = \mathrm{Bdiag}(W, I_p), \quad \textit{and} \quad \hat{V} = \mathrm{Bdiag}(V, I_m).$$

Proof. From (13.19) and (13.21), it is clear that

$$(A_{\rm red}, B_{\rm red}, C_{\rm red}, D_{\rm red}) \in \Sigma_{\mathcal{D}}^{\rm red}(W,V)$$

if and only if

$$\begin{bmatrix} A_{\rm red} & B_{\rm red} \\ C_{\rm red} & D_{\rm red} \end{bmatrix}^\top \in \hat{V}^\top \mathcal{Z}_{n+m}(N)\hat{W}. \tag{13.23}$$

Therefore, what needs to be proven is $\mathcal{Z}_{r+m}(N_{\rm red}(W,V)) = \hat{V}^\top \mathcal{Z}_{n+m}(N)\hat{W}$. To do so, first observe that both N_{22} and $N \mid N_{22}$ are nonsingular since $N \in \mathbf{\Pi}_{n+p,n+m}$ is nonsingular.

Now, let $Z \in \mathcal{Z}_{n+m}(N)$. From (A.9), we see that

$$N \mid N_{22} + (Z + N_{22}^{-1}N_{21})^\top N_{22}(Z + N_{22}^{-1}N_{21}) \geqslant 0$$

or equivalently, by a Schur complement argument,

$$\begin{bmatrix} N \mid N_{22} & (Z + N_{22}^{-1}N_{21})^\top \\ Z + N_{22}^{-1}N_{21} & -N_{22}^{-1} \end{bmatrix} \geqslant 0.$$

By post- and pre-multiplying the matrix above by $\mathrm{Bdiag}(\hat{W},\hat{V})$ and its transpose, respectively, we obtain

$$\begin{bmatrix} \hat{W}^\top (N \mid N_{22})\hat{W} & \hat{W}^\top (Z + N_{22}^{-1}N_{21})^\top \hat{V} \\ \hat{V}^\top (Z + N_{22}^{-1}N_{21})\hat{W} & -\hat{V}^\top N_{22}^{-1}\hat{V} \end{bmatrix} \geqslant 0.$$

By taking the Schur complement again, we see that $\hat{V}^\top Z\hat{W} \in \mathcal{Z}_{r+m}(N_{\rm red}(W,V))$. This proves that $\hat{V}^\top \mathcal{Z}_{n+m}(N)\hat{W} \subseteq \mathcal{Z}_{r+m}(N_{\rm red}(W,V))$.

To show the reverse inclusion, let $\hat{Z} \in \mathcal{Z}_{r+m}(N_{\rm red}(W,V))$. It follows from Theorem A.6.a that

$$\hat{Z} = -\hat{V}^\top N_{22}^{-1}N_{21}\hat{W} + (-\hat{V}^\top N_{22}^{-1}\hat{V})^{\frac{1}{2}} \hat{S}\hat{W}^\top (N \mid N_{22})\hat{W} \tag{13.24}$$

for some $\hat{S} \in \mathbb{R}^{(r+m)\times(r+p)}$ with $\hat{S}^\top \hat{S} \leqslant I_{r+p}$. Now, note that

$$(-\hat{V}^\top N_{22}^{-1}\hat{V})^{\frac{1}{2}}(-\hat{V}^\top N_{22}^{-1}\hat{V})^{\frac{1}{2}} = -\hat{V}^\top N_{22}^{-1}\hat{V} = \hat{V}^\top((-N_{22})^{-1})^{\frac{1}{2}}((-N_{22})^{-1})^{\frac{1}{2}}\hat{V}$$

and

$$\hat{W}^\top (N \,|\, N_{22}) \hat{W} = \hat{W}^\top (N \,|\, N_{22})^{\frac{1}{2}} (N \,|\, N_{22})^{\frac{1}{2}} \hat{W}.$$

Then, Lemma A.1.(a) implies that there exist $S_1 \in \mathbb{R}^{(n+m)\times(r+m)}$ and $S_2 \in \mathbb{R}^{(r+p)\times(n+p)}$ with $S_1^\top S_1 \leqslant I_{r+m}$ and $S_2^\top S_2 \leqslant I_{n+p}$ such that

$$(-\hat{V}^\top N_{22}^{-1} \hat{V})^{\frac{1}{2}} = \hat{V}^\top ((-N_{22})^{-1})^{\frac{1}{2}} S_1 \quad \text{and} \quad \hat{W}^\top (N \,|\, N_{22}) \hat{W} = S_2 (N \,|\, N_{22})^{\frac{1}{2}} \hat{W}.$$

By substituting into (13.24), we get

$$\hat{Z} = \hat{V}^\top \left(-N_{22}^{-1} N_{21} + ((-N_{22})^{-1})^{\frac{1}{2}} S_1 \hat{S} S_2 (N \,|\, N_{22})^{\frac{1}{2}} \right) \hat{W}.$$

Let $S = S_1 \hat{S} S_2$. Note that $S^\top S \leqslant I_{n+m}$ since $S_1^\top S_1 \leqslant I_{r+m}$, $\hat{S}^\top \hat{S} \leqslant I_{r+p}$, and $S_2^\top S_2 \leqslant I_{n+p}$. Therefore, we see from Theorem A.6.a that

$$-N_{22}^{-1} N_{21} + ((-N_{22})^{-1})^{\frac{1}{2}} S_1 \hat{S} S_2 (N \,|\, N_{22})^{\frac{1}{2}} \in \mathcal{Z}_{n+m}(N).$$

Consequently, $\hat{Z} \in \hat{V}^\top \mathcal{Z}_{n+m}(N) \hat{W}$ proving that

$$\mathcal{Z}_{r+m}(N_{\mathrm{red}}(W, V)) \subseteq \hat{V}^\top \mathcal{Z}_{n+m}(N) \hat{W}.$$

□

Theorem 13.4 has a nice interpretation in terms of *data reduction*. Namely, the matrix $N_{\mathrm{red}}(W, V)$ characterizing all reduced-order models depends only on the matrices $\hat{V}, \hat{W}$ and the original data matrix N. As such, $N_{\mathrm{red}}(W, V)$ is constructed from the data and noise model only. Importantly, $N_{\mathrm{red}}(W, V)$ has a lower dimension than N and can thus be regarded as a reduced data matrix. Hence, we can characterize all reduced-order models by directly reducing the data matrix N rather than reducing individual systems $(A, B, C, D) \in \Sigma_{\mathcal{D}}$. It is also worth mentioning that

$$(W^\top A_{\mathrm{true}} V, W^\top B_{\mathrm{true}}, C_{\mathrm{true}} V, D_{\mathrm{true}}) \in \Sigma_{\mathcal{D}}^{\mathrm{red}}(W, V).$$

In this section, reduced-order approximations of data-consistent systems were obtained from the collected data by employing *given* Petrov-Galerkin projection matrices V and W. In the next section, we will construct the Petrov-Galerkin projection matrices on the basis of the available data via generalized balancing.

13.4 Informativity for generalized Lyapunov balancing

First, we introduce the notion of informativity for generalized Lyapunov balancing (GLB).

Definition 13.5. We say that the data (U_-, X, Y_-) are *informative for GLB* if there exist $\hat{P} > 0$ and $\hat{Q} > 0$ such that

$$\hat{P} - A\hat{P}A^\top > BB^\top \tag{13.25a}$$

and

$$\hat{Q} - A^\top\hat{Q}A > C^\top C \tag{13.25b}$$

for all $(A, B, C, D) \in \Sigma_\mathcal{D}$.

Stated differently, informativity for GLB requires the existence of *common* generalized controllability and observability Gramians for all data-consistent systems. To present necessary and sufficient conditions for informativity for GLB, we first define

$$N_\mathcal{C} := \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ 0 & I_{n+m} \end{bmatrix}^\top N \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ 0 & I_{n+m} \end{bmatrix}$$

and

$$N_\mathcal{O} := \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ 0 & I_{n+p} \end{bmatrix}^\top N^\star \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ 0 & I_{n+p} \end{bmatrix}$$

where

$$N^\star := \begin{bmatrix} 0 & -I_{n+m} \\ I_{n+p} & 0 \end{bmatrix} N^{-1} \begin{bmatrix} 0 & -I_{n+p} \\ I_{n+m} & 0 \end{bmatrix}.$$

Theorem 13.6. *The data (U_-, X, Y_-) are informative for GLB if and only if there exist $\hat{P} > 0$, $\hat{Q} > 0$, and scalars $\alpha, \beta > 0$ such that*

$$\begin{bmatrix} \hat{P} & 0 & 0 \\ 0 & -\hat{P} & 0 \\ 0 & 0 & -I_m \end{bmatrix} - \alpha N_\mathcal{C} > 0 \tag{13.26a}$$

and

$$\begin{bmatrix} \hat{Q} & 0 & 0 \\ 0 & -\hat{Q} & 0 \\ 0 & 0 & -I_p \end{bmatrix} - \beta N_\mathcal{O} > 0. \tag{13.26b}$$

Proof. We first claim that the existence of $\hat{P} > 0$ satisfying (13.25a) for all $(A, B, C, D) \in \Sigma_\mathcal{D}$ is equivalent to the existence of $\hat{P} > 0$ and $\alpha > 0$ satisfying (13.26a). To see this, note first that (13.25a) is equivalent to the quadratic matrix inequality

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} \hat{P} & 0 & 0 \\ 0 & -\hat{P} & 0 \\ 0 & 0 & -I_m \end{bmatrix} \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} > 0. \tag{13.27}$$

Next, recall from (13.19) that $(A, B, C, D) \in \Sigma_{\mathcal{D}}$ if and only if

$$\begin{bmatrix} A^\top & C^\top \\ B^\top & D^\top \end{bmatrix} \in \mathcal{Z}_{n+m}(N). \tag{13.28}$$

Therefore, there exists (C, D) such that $(A, B, C, D) \in \Sigma_{\mathcal{D}}$ if and only if

$$\begin{bmatrix} A^\top \\ B^\top \end{bmatrix} \in \mathcal{Z}_{n+m}(N) \begin{bmatrix} I_n \\ 0_{p,n} \end{bmatrix} = \mathcal{Z}_{n+m}(N_{\mathcal{C}})$$

where the last equality follows from Theorem A.7. By applying Theorem A.17, we see that there exists $\hat{P} > 0$ such that (13.25a) is satisfied for all $(A, B, C, D) \in \Sigma_{\mathcal{D}}$ if and only if there exist $\hat{P} > 0$ and $\alpha \geqslant 0$ such that (13.26a) is satisfied. Because of the $-I_m$ term in (13.26a), it is clear that α has to be nonzero.

Now, we claim that the existence of a matrix $\hat{Q} > 0$ satisfying (13.25b) for all data-consistent systems $(A, B, C, D) \in \Sigma_{\mathcal{D}}$ is equivalent to the existence of $\hat{Q} > 0$ and $\beta > 0$ satisfying (13.26b). To see this, first note that (13.25b) is equivalent to the quadratic matrix inequality

$$\begin{bmatrix} I \\ A \\ C \end{bmatrix}^\top \begin{bmatrix} \hat{Q} & 0 & 0 \\ 0 & -\hat{Q} & 0 \\ 0 & 0 & -I_p \end{bmatrix} \begin{bmatrix} I \\ A \\ C \end{bmatrix} > 0. \tag{13.29}$$

In addition, note that (13.28) and Theorem A.3 imply that $(A, B, C, D) \in \Sigma_{\mathcal{D}}$ if and only if

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{Z}_{n+p}(N^\star).$$

Therefore, there exists (B, D) such that $(A, B, C, D) \in \Sigma_{\mathcal{D}}$ if and only if

$$\begin{bmatrix} A \\ C \end{bmatrix} \in \mathcal{Z}_{n+p}(N^\star) \begin{bmatrix} I_n \\ 0_{m,n} \end{bmatrix} = \mathcal{Z}_{n+p}(N_{\mathcal{O}})$$

where the last equality follows from Theorem A.7. By applying Theorem A.17, we see that there exists $\hat{Q} > 0$ such that (13.25b) is satisfied for all $(A, B, C, D) \in \Sigma_{\mathcal{D}}$ if and only if there exist $\hat{Q} > 0$ and $\beta \geqslant 0$ such that (13.26b) is satisfied. Finally, β has to be nonzero because of the $-I_p$ term in (13.26b). □

13.5 Reduced-order models and error analysis

A direct consequence of data informativity for GLB is that all data-consistent systems are stable and share the *common* generalized Gramians $\hat{P}$ and $\hat{Q}$. As a result, all such systems can be balanced by a *common* balancing transformation matrix $\hat{S}$ satisfying $\hat{S}\hat{P}\hat{S}^\top = \hat{S}^{-\top}\hat{Q}\hat{S}^{-1} = \hat{H}_{\mathrm{sv}}$ where $\hat{H}_{\mathrm{sv}}$ is a matrix of the form (13.6), containing the common generalized Hankel singular values. Further, the Petrov-Galerkin projection

$$W = \hat{S}^\top \begin{bmatrix} I_r \\ 0 \end{bmatrix} \quad \text{and} \quad V = \hat{S}^{-1} \begin{bmatrix} I_r \\ 0 \end{bmatrix}. \tag{13.30}$$

can be applied to each data-compatible system to obtain a corresponding reduced-order model as in (13.3). This leads to the set of reduced-order models

$$\Sigma_{\mathcal{D}}^{\mathrm{red}} := \left\{ (W^\top AV, W^\top B, CV, D) \mid (A, B, C, D) \in \Sigma_{\mathcal{D}} \right\}. \tag{13.31}$$

As we have seen earlier in Theorem 13.4, the set of reduced-order models is a set induced by a quadratic matrix inequality.

Suppose that $(A, B, C, D) \in \Sigma_{\mathcal{D}}$. Let $(A_{\mathrm{red}}, B_{\mathrm{red}}, C_{\mathrm{red}}, D_{\mathrm{red}})$ denote the corresponding reduced-order system in $\Sigma_{\mathcal{D}}^{\mathrm{red}}$. We know from Proposition 13.2 that the $\mathcal{H}_\infty$-norm error between the original and reduced-order systems is upper bounded by the neglected common generalized Hankel singular values, that is

$$\|G - G_{\mathrm{red}}\|_{\mathcal{H}_\infty} \leqslant 2 \sum_{i=k+1}^{\hat{\kappa}} \hat{\sigma}_i \tag{13.32}$$

provided that $r = \sum_{i=1}^{k} \hat{m}_i$ with $k < \hat{\kappa}$. This error bound allows us to evaluate the quality of the reduced-order model obtained from a *specific* data-consistent system. Since the true system is *unknown* and could be any data-consistent system within $\Sigma_{\mathcal{D}}$, obtaining its corresponding reduced-order model is not possible and any reduced-order model within $\Sigma_{\mathcal{D}}^{\mathrm{red}}$ can serve as an approximation of the true system. As such, the error bound (13.32) provides very limited insight in the error analysis. Instead, the quality of the reduced-order model $(\hat{A}, \hat{B}, \hat{C}, \hat{D}) \in \Sigma_{\mathcal{D}}^{\mathrm{red}}$ as an approximation of the true system $(A_{\mathrm{true}}, B_{\mathrm{true}}, C_{\mathrm{true}}, D_{\mathrm{true}})$ is determined by the error:

$$\|G_{(A_{\mathrm{true}}, B_{\mathrm{true}}, C_{\mathrm{true}}, D_{\mathrm{true}})} - G_{(\hat{A}, \hat{B}, \hat{C}, \hat{D})}\|_{\mathcal{H}_\infty} \tag{13.33}$$

where

$$G_{(A,B,C,D)} = D + C(sI - A)^{-1}B.$$

In what follows, we will investigate two error bounds:

- an *a priori* bound that accounts for any data-consistent system and any reduced-order model,
- an *a posteriori* bound that considers any data-consistent system but focuses on a specific choice of the reduced-order model.

A priori error bound

An *a priori* upper bound on the error can be obtained by computing the maximum possible error between a data-consistent system and a reduced-order model. The following theorem provides LMIs to check whether such an upper bound is less than a given number.

Theorem 13.7. *The error bound*

$$\|G_{(A,B,C,D)} - G_{(\hat{A},\hat{B},\hat{C},\hat{D})}\|_{\mathcal{H}_\infty} < \gamma \tag{13.34}$$

holds for every $(A,B,C,D) \in \Sigma_{\mathcal{D}}$ *and every* $(\hat{A},\hat{B},\hat{C},\hat{D}) \in \Sigma_{\mathcal{D}}^{\mathrm{red}}$ *if there exist a positive definite matrix* $K \in \mathbb{S}^{n+r}$*, and scalars* $\delta > 0$*,* $\eta > 0$*, and* μ *such that*

$$\Theta(K,\mu) - \begin{bmatrix} \delta N & 0 \\ 0 & \eta N_{\mathrm{red}} \end{bmatrix} > 0 \tag{13.35}$$

where

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \text{ with } K_{11} \in \mathbb{S}^n,$$

$$\Theta(K,\mu) = \begin{bmatrix}
K_{11} & 0 & 0 & 0 & K_{12} & 0 & 0 & 0 \\
0 & (\frac{1}{2}-\mu)I_p & 0 & 0 & 0 & -\mu I_p & 0 & 0 \\
0 & 0 & -K_{11} & 0 & 0 & 0 & -K_{12} & 0 \\
0 & 0 & 0 & -\gamma^{-2}I_m & 0 & 0 & 0 & -\gamma^{-2}I_m \\
K_{21} & 0 & 0 & 0 & K_{22} & 0 & 0 & 0 \\
0 & -\mu I_p & 0 & 0 & 0 & (\frac{1}{2}-\mu)I_p & 0 & 0 \\
0 & 0 & -K_{21} & 0 & 0 & 0 & -K_{22} & 0 \\
0 & 0 & 0 & -\gamma^{-2}I_m & 0 & 0 & 0 & -\gamma^{-2}I_m
\end{bmatrix},$$

and N *and* N_{red} *are given by* (13.18) *and* (13.22)*, respectively.*

Proof. Let $(A,B,C,D) \in \Sigma_{\mathcal{D}}$ and $(\hat{A},\hat{B},\hat{C},\hat{D}) \in \Sigma_{\mathcal{D}}^{\mathrm{red}}$. Then, we have

$$\begin{bmatrix} I & 0 \\ 0 & I \\ A^\top & C^\top \\ B^\top & D^\top \end{bmatrix}^\top N \begin{bmatrix} I & 0 \\ 0 & I \\ A^\top & C^\top \\ B^\top & D^\top \end{bmatrix} \geqslant 0 \tag{13.36a}$$

$$\begin{bmatrix} I & 0 \\ 0 & I \\ \hat{A}^\top & \hat{C}^\top \\ \hat{B}^\top & \hat{D}^\top \end{bmatrix}^\top N_{\mathrm{red}} \begin{bmatrix} I & 0 \\ 0 & I \\ \hat{A}^\top & \hat{C}^\top \\ \hat{B}^\top & \hat{D}^\top \end{bmatrix} \geqslant 0. \tag{13.36b}$$

It follows from (13.35) and (13.36) that

$$\Lambda^\top \Theta(K,\mu)\Lambda > 0 \tag{13.37}$$

where

$$\Lambda = \text{Bdiag}\left(\begin{bmatrix} I & 0 \\ 0 & I \\ A^\top & C^\top \\ B^\top & D^\top \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & I \\ \hat{A}^\top & \hat{C}^\top \\ \hat{B}^\top & \hat{D}^\top \end{bmatrix} \right).$$

Observe that

$$\Lambda \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_p \\ 0 & I_r & 0 \\ 0 & 0 & -I_p \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_p \\ A^\top & 0 & C^\top \\ B^\top & 0 & D^\top \\ 0 & I_r & 0 \\ 0 & 0 & -I_p \\ 0 & \hat{A}^\top & -\hat{C}^\top \\ 0 & \hat{B}^\top & -\hat{D}^\top \end{bmatrix}. \tag{13.38}$$

Then, it follows from (13.37) and (13.38) that

$$\begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_p \\ A^\top & 0 & C^\top \\ B^\top & 0 & D^\top \\ 0 & I_r & 0 \\ 0 & 0 & -I_p \\ 0 & \hat{A}^\top & -\hat{C}^\top \\ 0 & \hat{B}^\top & -\hat{D}^\top \end{bmatrix}^\top \Theta(K,\mu) \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_p \\ A^\top & 0 & C^\top \\ B^\top & 0 & D^\top \\ 0 & I_r & 0 \\ 0 & 0 & -I_p \\ 0 & \hat{A}^\top & -\hat{C}^\top \\ 0 & \hat{B}^\top & -\hat{D}^\top \end{bmatrix} > 0. \tag{13.39}$$

Straightforward calculations yield that

$$\begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_p \\ A^\top & 0 & C^\top \\ B^\top & 0 & D^\top \\ 0 & I_r & 0 \\ 0 & 0 & -I_p \\ 0 & \hat{A}^\top & -\hat{C}^\top \\ 0 & \hat{B}^\top & -\hat{D}^\top \end{bmatrix}^\top \Theta(K,\mu) \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_p \\ A^\top & 0 & C^\top \\ B^\top & 0 & D^\top \\ 0 & I_r & 0 \\ 0 & 0 & -I_p \\ 0 & \hat{A}^\top & -\hat{C}^\top \\ 0 & \hat{B}^\top & -\hat{D}^\top \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} K & 0 \\ 0 & I_p \end{bmatrix} - \begin{bmatrix} A_\text{e} & B_\text{e} \\ C_\text{e} & D_\text{e} \end{bmatrix} \begin{bmatrix} K & 0 \\ 0 & \gamma^{-2} I_m \end{bmatrix} \begin{bmatrix} A_\text{e} & B_\text{e} \\ C_\text{e} & D_\text{e} \end{bmatrix}^\top$$

where

$$A_e := \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad B_e := \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C_e := \begin{bmatrix} C & -\hat{C} \end{bmatrix}, \quad \text{and} \quad D_e := D - \hat{D}.$$

Hence, we see from (13.39) that

$$\begin{bmatrix} K & 0 \\ 0 & I_p \end{bmatrix} - \begin{bmatrix} A_e & B_e \\ C_e & D_e \end{bmatrix} \begin{bmatrix} K & 0 \\ 0 & \gamma^{-2} I_m \end{bmatrix} \begin{bmatrix} A_e & B_e \\ C_e & D_e \end{bmatrix}^\top > 0$$

which is, by a Schur complement argument, equivalent to

$$\begin{bmatrix} K & 0 & A_e & B_e \\ 0 & I_p & C_e & D_e \\ A_e^\top & C_e^\top & K^{-1} & 0 \\ B_e^\top & D_e^\top & 0 & \gamma^2 I_m \end{bmatrix} > 0.$$

By taking the Schur complement with respect to $\begin{bmatrix} K & 0 \\ 0 & I \end{bmatrix}$, we see that the last inequality is equivalent to

$$\begin{bmatrix} K^{-1} & 0 \\ 0 & \gamma^2 I_m \end{bmatrix} - \begin{bmatrix} A_e & B_e \\ C_e & D_e \end{bmatrix}^\top \begin{bmatrix} K^{-1} & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} A_e & B_e \\ C_e & D_e \end{bmatrix} > 0.$$

Therefore, it follows from Proposition 8.11 that A_e is stable and

$$\|G_{(A_e,B_e,C_e,D_e)}\|_{\mathcal{H}_\infty} < \gamma.$$

Finally, the observation

$$G_{(A_e,B_e,C_e,D_e)} = G_{(A,B,C,D)} - G_{(\hat{A},\hat{B},\hat{C},\hat{D})}$$

concludes the proof. □

A posteriori error bound

The *a priori* error bound that is provided by Theorem 13.7 is valid for *any* reduced-order model. The following result deals with an *a posteriori* error bound for a *given* reduced-order model.

Theorem 13.8. *Let* $(\hat{A}, \hat{B}, \hat{C}, \hat{D}) \in \Sigma_{\mathcal{D}}^{\text{red}}$ *and* $\gamma > 0$ *be a scalar. Then, the error bound*

$$\|G_{(A,B,C,D)} - G_{(\hat{A},\hat{B},\hat{C},\hat{D})}\|_{\mathcal{H}_\infty} < \gamma_0 \tag{13.40}$$

holds for every $(A,B,C,D) \in \Sigma_{\mathcal{D}}$ *if there exist a positive definite matrix* $K \in \mathbb{S}^{n+r}$ *and a scalar* $\delta > 0$ *such that*

$$\left[\begin{array}{cccc:c} K_{11} & 0 & 0 & 0 & K_{12} \\ 0 & I_p - \hat{C}K_{22}\hat{C}^\top - \gamma_0^{-2}\hat{D}\hat{D}^\top & \hat{C}K_{21} & \gamma_0^{-2}\hat{D} & \hat{C}K_{22}\hat{A}^\top + \gamma_0^{-2}\hat{D}\hat{B}^\top \\ 0 & K_{12}\hat{C}^\top & -K_{11} & 0 & -K_{12}\hat{A}^\top \\ 0 & \gamma_0^{-2}\hat{D}^\top & 0 & -\gamma_0^{-2}I_m & -\gamma_0^{-2}\hat{B}^\top \\ \hdashline K_{21} & \hat{A}K_{22}\hat{C}^\top + \gamma_0^{-2}\hat{B}\hat{D}^\top & -\hat{A}K_{21} & -\gamma_0^{-2}\hat{B} & K_{22} - \hat{A}K_{22}\hat{A}^\top - \gamma_0^{-2}\hat{B}\hat{B}^\top \end{array}\right] - \left[\begin{array}{c:c} \delta N & 0 \\ \hdashline 0 & 0 \end{array}\right] > 0 \tag{13.41}$$

where

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \text{ with } K_{11} \in \mathbb{S}^n$$

and N *is given by* (13.18).

Proof. Let $(A,B,C,D) \in \Sigma_{\mathcal{D}}$. Then, we have

$$\begin{bmatrix} I & 0 \\ 0 & I \\ A^\top & C^\top \\ B^\top & D^\top \end{bmatrix}^\top N \begin{bmatrix} I & 0 \\ 0 & I \\ A^\top & C^\top \\ B^\top & D^\top \end{bmatrix} \geqslant 0. \tag{13.42}$$

Let Θ denote the matrix on the left hand side of the inequality in (13.41). It follows from (13.42), (13.41), and straightforward calculations that

$$0 < \begin{bmatrix} I & 0 & A & B & 0 \\ 0 & 0 & 0 & 0 & I_r \\ 0 & I & C & D & 0 \end{bmatrix} \Theta \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ A^\top & 0 & C^\top \\ B^\top & 0 & D^\top \\ 0 & I_r & 0 \end{bmatrix} = \begin{bmatrix} K & 0 \\ 0 & I_p \end{bmatrix} - \Omega \begin{bmatrix} K & 0 \\ 0 & \gamma_0^{-2}I_m \end{bmatrix} \Omega^\top$$

where

$$\Omega = \begin{bmatrix} A_\mathrm{e} & B_\mathrm{e} \\ C_\mathrm{e} & D_\mathrm{e} \end{bmatrix}$$

with

$$A_\mathrm{e} := \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad B_\mathrm{e} := \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C_\mathrm{e} := \begin{bmatrix} C & -\hat{C} \end{bmatrix}, \quad \text{and} \quad D_\mathrm{e} := D - \hat{D}.$$

The arguments employed in the proof of Theorem 13.7 together with Proposition 13.7 imply that A_e is stable and $\|G_{(A_\mathrm{e},B_\mathrm{e},C_\mathrm{e},D_\mathrm{e})}\|_{\mathcal{H}_\infty} < \gamma_0$. Finally, the observation

$$G_{(A_\mathrm{e},B_\mathrm{e},C_\mathrm{e},D_\mathrm{e})} = G_{(A,B,C,D)} - G_{(\hat{A},\hat{B},\hat{C},\hat{D})}.$$

concludes the proof. □

13.6 Illustrative example

To illustrate the results presented so far in this chapter, we consider the discrete-time system of the form (13.13) where

$$A_{\text{true}} = \begin{bmatrix} 0.9299 & 0.4160 & 0.7447 & 0.2291 & 0.2452 & 0.0592 \\ -0.1869 & 0.7430 & 0.3318 & 0.7617 & 1.0859 & 0.3560 \\ 0.0380 & 0.0477 & -0.3644 & 0.0647 & 0.1370 & 0.0766 \\ 0.0169 & 0.0549 & -0.0972 & -0.3693 & -0.8685 & 0.0484 \\ 0.0250 & 0.0285 & 0.2741 & 0.1393 & -0.0474 & 0.1615 \\ 0.1108 & 0.1358 & -1.7370 & 0.1855 & -1.8002 & -0.2311 \end{bmatrix},$$

$$B_{\text{true}} = \begin{bmatrix} 0.0701 \\ 0.1869 \\ -0.0380 \\ -0.0169 \\ -0.0250 \\ -0.1108 \end{bmatrix}, C_{\text{true}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } D_{\text{true}} = 0.$$

This system is obtained as the zero-order hold discretization (with sampling time 0.5 seconds) of the continuous-time state-space model of a cart with double pendulum presented in [78].

To illustrate the data-driven model reduction from noisy data, we apply the input signal

$$u(t) = 2\sin(t) + \cos(0.5t) \tag{13.43}$$

to the system (13.13) and collect $T = 200$ data samples for initial states and that were drawn randomly from a Gaussian distribution with zero mean and unit variance. Also the noise samples were drawn randomly from a Gaussian distribution, with zero mean and variance σ^2. In this example, we assume knowledge of a bound on the energy of the noise which corresponds to the noise model (13.15) with $\Phi_{11} = 1.35\sigma^2 I$, $\Phi_{12} = 0$ and $\Phi_{22} = -I$.

We simulated the noise with different levels: $\sigma \in \{0.002, 0.005, 0.01, 0.03, 0.05\}$ and verified that the noise model is satisfied by the generated noise sequences. In addition, Assumption 13.3 is satisfied by the collected data together with the noise model.

For each noise level, we applied Theorem 13.3 to the collected data set and observed that each data set is informative for generalized Lyapunov balancing. In Figure 13.1, the generalized Hankel singular values obtained from each data set are depicted. As expected, the generalized Hankel singular values provide less strict bounds on the unknown ordinary Hankel singular values when the noise level is increased.

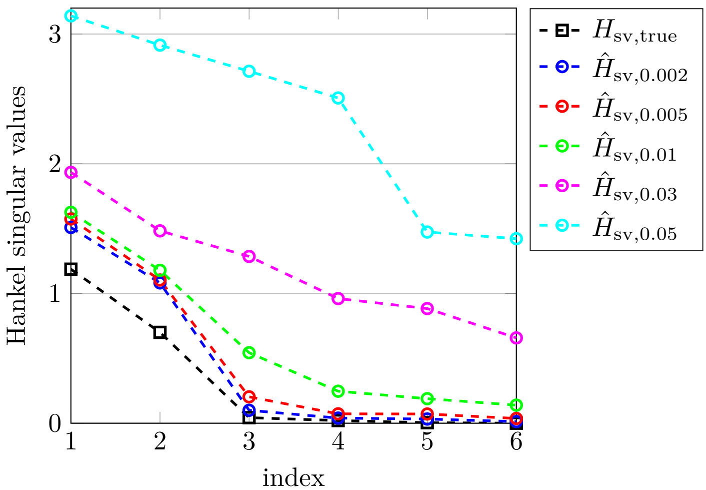


Figure 13.1: The Hankel singular values of the true system denoted by $H_{\mathrm{sv,true}}$ and generalized Hankel singular values of all data-consistent systems for different noise levels denoted by $\hat{H}_{\mathrm{sv},\sigma}$.

We picked a reduced-order model of order $r = 3$ from the set $\Sigma_{\mathcal{D}}^{\mathrm{red}}$ for each data set. The Bode diagrams of the reduced-order systems are depicted in Figure 13.2. This figure shows that reduced-order models accurately approximate the true system at least up to the noise level $\sigma = 0.03$ while the resulting reduced-order model for $\sigma = 0.05$ approximates the true system poorly. Figures 13.3-13.4 depict the output trajectories of the true system and the reduced-order model obtained from the data corresponding to the noise level $\sigma = 0.03$ and the error.

By applying Theorems 13.7-13.8 and the bisection method, we obtained best a priori and a posteriori error bounds for each data set. Figure 13.5 shows how these error bounds vary depending on the noise level. As expected, these bounds are getting more conservative when the noise level increases. It is also clear that the a posteriori upper bound is less conservative than the corresponding a priori upper bound for each noise level. In spite of the conservative error bounds, the actual $\mathcal{h}_\infty$-norms of the errors between the true system and the reduced-order models for some small enough noise levels show that the proposed data-driven method performs well. In particular, the $\mathcal{h}_\infty$-norm of the errors for noise levels $\sigma = 0.002, 0.005, 0.01$ and 0.03 which are given by 0.0405, 0.0470, 0.0507 and

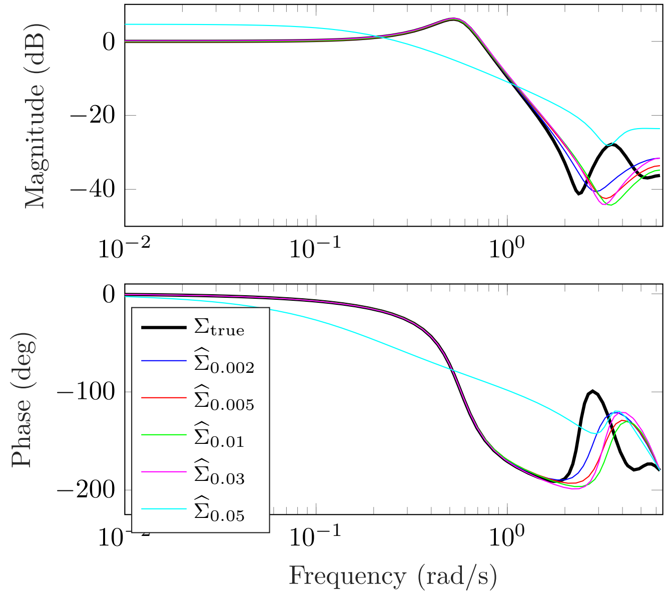


Figure 13.2: Bode plots of the true system Σ_{true} and reduced-order models $\widehat{\Sigma}_\sigma$ for five different noise levels $\sigma = 0.002, 0.005, 0.01, 0.03$, and 0.05.

0.0513, respectively, are relatively small compared to the error of reduction by the ordinary balanced truncation, which is equal to 0.0314.

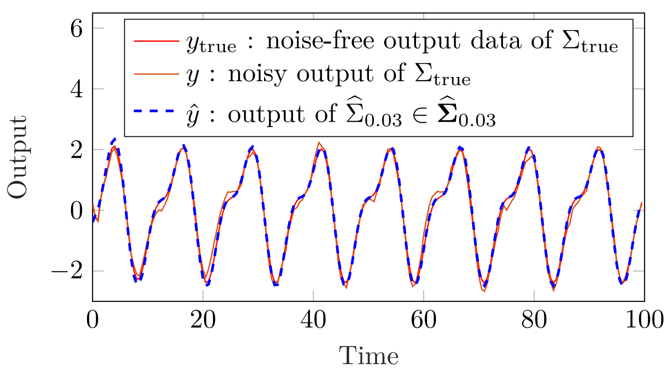


Figure 13.3: Comparison of output trajectories.

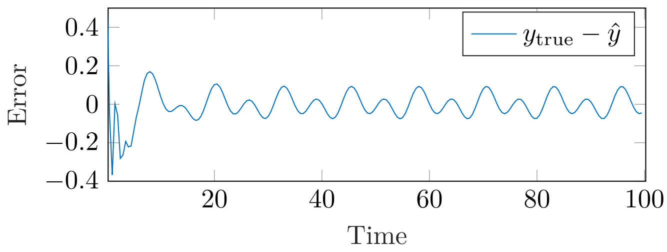


Figure 13.4: Error $y_{\text{true}} - \hat{y}$.

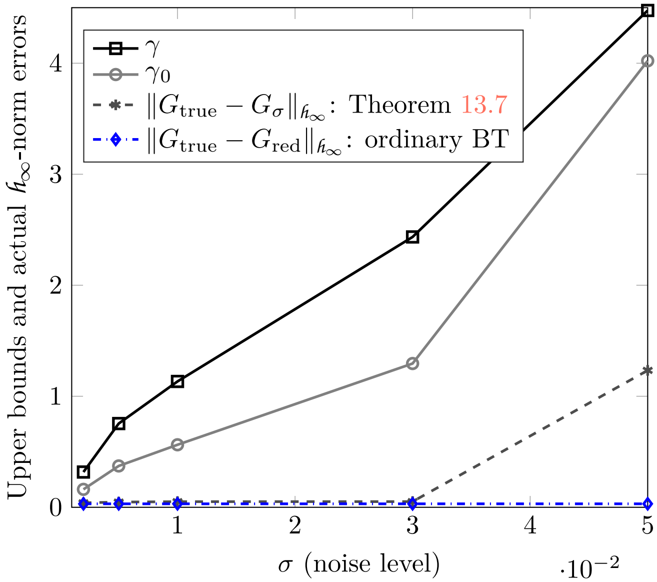


Figure 13.5: Comparison of a priori error bounds γ, a posteriori error bounds γ_0, the actual $\mathcal{H}_\infty$-norm of the error $G_{\text{true}} - G_\sigma$ where G_σ is the transfer matrix of a reduced-order model obtained using Theorem 13.7, and $\mathcal{H}_\infty$-norm of the error $G_{\text{true}} - G_{\text{red}}$ where G_{red} is the transfer matrix of the reduced-order model obtained via ordinary balanced truncation.

13.7 Notes and references

The Lyapunov balancing method was first introduced in [118] and later in the systems and control literature in [116]. We refer the reader to the excellent survey paper [67] for a historical and detailed treatment. Various approaches to data-driven Lyapunov balancing have been proposed in the literature. Examples include [105, 135] that propose a data-driven balanced truncation method from *persistently exciting* data and [66] that estimates Gramians from frequency and time-domain data based on their quadrature form. Existing methods for data-driven model reduction do not provide conditions on the data that would guarantee preservation of system properties such as asymptotic stability. In addition, they do not provide error bounds, especially when the available data is subject to noise.

The results of this chapter are based on the paper [29] where the informativity for generalized Lyapunov balancing was studied the first time.

14

Data-driven model reduction via moment matching

In this chapter, we focus on interpolatory model reduction techniques. Together with the balancing methods as discussed in the previous chapter, the interpolatory methods form a popular class of model reduction approaches since they are numerically stable and, therefore, applicable to models of very large order. These methods aim at constructing a reduced-order model whose transfer function interpolates that of the original high-order model at selected interpolation points, e.g., [10]. The central problem in this chapter is to derive informativity conditions on the input-output data for moment matching as well as to develop methods to determine reduced-order models once the data are informative.

14.1 Single-input single-output AR models and data

Consider the discrete-time input-output system given by the autoregressive model of the form

$$\begin{aligned} y(t+n) + \bar{p}_{n-1}y(t+n-1) + \cdots + \bar{p}_1 y(t+1) + \bar{p}_0 y(t) \\ = \bar{q}_n u(t+n) + \bar{q}_{n-1}u(t+n-1) + \cdots + \bar{q}_1 u(t+1) + \bar{q}_0 u(t) \end{aligned} \tag{14.1}$$

where $n \in \mathbb{Z}_+$, u denotes the scalar input, y the scalar output. We refer this system as the *true* system and collect its scalar parameters $\bar{p}_i$ and $\bar{q}_i$ in the vectors

$$\bar{p} = \begin{bmatrix} \bar{p}_0 & \bar{p}_1 & \cdots & \bar{p}_{n-1} \end{bmatrix} \in \mathbb{R}^{1\times n}$$

and

$$\bar{q} = \begin{bmatrix} \bar{q}_0 & \bar{q}_1 & \cdots & \bar{q}_n \end{bmatrix} \in \mathbb{R}^{1\times (n+1)}.$$

We assume that $n \geqslant 0$ is *known*, the parameters $\begin{bmatrix} \bar{q} & -\bar{p} \end{bmatrix}$ are *unknown*, and *input-output data* $(u_{[0,T-1]}, y_{[0,T-1]})$, generated by the true system (14.1), are *given* for some $T \geqslant n+1$. We define

$$U = U_{[0,T-1]} \quad \text{and} \quad Y = Y_{[0,T-1]}.$$

Note that the data (U, Y) can be generated by a system of the form

$$\begin{aligned} & y(t+n) + p_{n-1}y(t+n-1) + \cdots + p_1 y(t+1) + p_0 y(t) \\ & \quad = q_n u(t+n) + q_{n-1}u(t+n-1) + \cdots + q_1 u(t+1) + q_0 u(t) \end{aligned} \tag{14.2}$$

if and only if

$$\begin{bmatrix} q & -p \end{bmatrix} \begin{bmatrix} H_{n+1}(u_{[0,T-1]}) \\ H_n(y_{[0,T-2]}) \end{bmatrix} = Y_{[n,T-1]} \tag{14.3}$$

where

$$\begin{aligned} p &= \begin{bmatrix} p_0 & p_1 & \cdots & p_{n-1} \end{bmatrix} \in \mathbb{R}^{1\times n} \\ q &= \begin{bmatrix} q_0 & q_1 & \cdots & q_n \end{bmatrix} \in \mathbb{R}^{1\times (n+1)}. \end{aligned}$$

Therefore, the set of all systems that are consistent with the data (U, Y) is given by

$$\Sigma_{(U,Y)} = \left\{ \begin{bmatrix} q & -p \end{bmatrix} \in \mathbb{R}^{1\times(2n+1)} \mid \text{(14.3) holds} \right\}. \tag{14.4}$$

Since the data (U, Y) are generated by the true system (14.1), we clearly have that

$$\begin{bmatrix} \bar{q} & -\bar{p} \end{bmatrix} \in \Sigma_{(U,Y)}.$$

An obvious question to ask is when the data uniquely determine the true system.

Definition 14.1. The data (U, Y) are *informative for system identification* if $\Sigma_{(U,Y)}$ is a singleton.

Data informativity for system identification can easily be characterized as follows.

Proposition 14.2. *The data (U, Y) are informative for system identification if and only if*

$$\operatorname{rank} \begin{bmatrix} H_{n+1}(u_{[0,T-1]}) \\ H_n(y_{[0,T-2]}) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} H_{n+1}(u_{[0,T-1]}) \\ H_{n+1}(y_{[0,T-1]}) \end{bmatrix} = 2n+1. \tag{14.5}$$

Proof. It is obvious that the first and the second equality in (14.5) are equivalent to the existence and uniqueness of the solution of the linear equation (14.3), respectively. □

14.2 The 0-th moment

Next, we will introduce the notion of the 0-th moment for systems of the form (14.2). To do so, we first recall the definition of the shift operator (see (9.2)), that is, $(\sigma f)(t) = f(t+1)$ for all $t \in \mathbb{Z}_+$. Then, (14.2) can be rewritten as

$$P(\sigma)y = Q(\sigma)u \tag{14.6}$$

where

$$P(\xi) = \xi^n + p_{n-1}\xi^{n-1} + \cdots + p_1\xi + p_0 \tag{14.7a}$$
$$Q(\xi) = q_n\xi^n + q_{n-1}\xi^{n-1} + \cdots + q_1\xi + q_0. \tag{14.7b}$$

Now, we are in a position to define the 0-th moment.

Definition 14.3 (0-th moment)**.** Given an interpolation point $\mu \in \mathbb{C}$, a scalar $M_0 \in \mathbb{C}$ is said to be a 0-*th moment at* μ of the discrete-time system (14.2) if

$$P(\mu)M_0 = Q(\mu). \tag{14.8}$$

Remark 14.4. For a discrete-time system (14.2) with transfer function

$$G(z) = \frac{Q(z)}{P(z)},$$

the 0-th moment at μ is typically defined as the complex number $Q(\mu)/P(\mu)$. This, however, requires that $P(\mu) \neq 0$. In other words, the 0-th moment is not defined when μ is a pole of $G(z)$. The notion in Definition 14.3 is a slight generalization as it allows to define a moment in case $P(\mu) = 0$ and/or $Q(\mu) = 0$. We stress that minimality is not assumed for (14.2). In case $P(\mu) = Q(\mu) = 0$, i.e. there is a pole-zero cancellation at μ, *any* complex number is regarded as a 0-th moment at μ by Definition 14.3.

14.3 Informativity for interpolation

Given an interpolation point $\mu \in \mathbb{C}$, we are interested in finding conditions on the data (U, Y) under which the 0-th moment at μ of the true *unknown* system can be computed. To derive such conditions, observe first that (14.8) is equivalent to

$$\begin{bmatrix} q & -p \end{bmatrix} \begin{bmatrix} \gamma_n(\mu) \\ M_0\gamma_{n-1}(\mu) \end{bmatrix} = M_0\mu^n \tag{14.9}$$

where

$$\gamma_\ell(\xi) = \begin{bmatrix} 1 \\ \xi \\ \vdots \\ \xi^\ell \end{bmatrix}. \tag{14.10}$$

Therefore, one can compute the 0-th moment at μ of the true system if and only if there exists a unique M_0 such that (14.9) is satisfied for any data-consistent system. This observation leads to the following informativity notion.

Definition 14.5. The data (U, Y) are *informative for interpolation at* μ if there exists a unique M_0 such that (14.9) holds for all $\begin{bmatrix} q & -p \end{bmatrix} \in \Sigma_{(U,Y)}$.

It is clear that such a unique M_0 exists if the data (U, Y) are informative for system identification as in Definition 14.1 and $P(\mu) \neq 0$. The following theorem shows, however, that the data can be informative for interpolation even if they are not so for system identification.

Theorem 14.6. *The data* (U, Y) *are informative for interpolation at* μ *if and only if*

$$\operatorname{rank} \begin{bmatrix} H_{n+1}(u_{[0,T-1]}) & 0 & \gamma_n(\mu) \\ H_{n+1}(y_{[0,T-1]}) & \gamma_n(\mu) & 0 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} H_{n+1}(u_{[0,T-1]}) & 0 \\ H_{n+1}(y_{[0,T-1]}) & \gamma_n(\mu) \end{bmatrix} \tag{14.11}$$

and

$$\operatorname{rank} \begin{bmatrix} H_{n+1}(u_{[0,T-1]}) & 0 \\ H_{n+1}(y_{[0,T-1]}) & \gamma_n(\mu) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} H_{n+1}(u_{[0,T-1]}) \\ H_{n+1}(y_{[0,T-1]}) \end{bmatrix} + 1. \tag{14.12}$$

To prove this theorem, we need the following two rather elementary linear algebra results.

Lemma 14.7. *Let* $A_i \in \mathbb{R}^{n \times m_i}$ *and* $b_i \in \mathbb{R}^{1 \times m_i}$ *for* $i = 1, 2$. *Consider the sets* $\mathcal{X}_i = \{\xi \in \mathbb{R}^{1 \times n} \mid \xi A_i = b_i\}$ *and assume that* $\mathcal{X}_1$ *is nonempty. Then,* $\mathcal{X}_1 \subseteq \mathcal{X}_2$ *if and only if*

$$\operatorname{im} \begin{bmatrix} A_2 \\ b_2 \end{bmatrix} \subseteq \operatorname{im} \begin{bmatrix} A_1 \\ b_1 \end{bmatrix}. \tag{14.13}$$

Proof. *if:* Suppose that (14.13) holds. Then, there exists F such that

$$\begin{bmatrix} A_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ b_1 \end{bmatrix} F.$$

This readily implies that $\mathcal{X}_1 \subseteq \mathcal{X}_2$.

only if: Suppose that $\mathcal{X}_1 \subseteq \mathcal{X}_2$. Let

$$\begin{bmatrix} \xi & \eta \end{bmatrix} \in \operatorname{lker} \begin{bmatrix} A_1 \\ b_1 \end{bmatrix} \tag{14.14}$$

where $\xi \in \mathbb{R}^{1 \times n}$ and $\eta \in \mathbb{R}$. We distinguish two cases:

Case 1: $\eta \neq 0$. By (14.14), we have $(-\xi/\eta)\, A_1 = b_1$, i.e., $-\xi/\eta \in \mathcal{X}_1$. Since $\mathcal{X}_1 \subseteq \mathcal{X}_2$, we also have that $(-\xi/\eta)\, A_2 = b_2$, which is equivalent to

$$\begin{bmatrix} \xi & \eta \end{bmatrix} \in \operatorname{lker} \begin{bmatrix} A_2 \\ b_2 \end{bmatrix}.$$

Case 2: $\eta = 0$. By (14.14), we have $\xi A_1 = 0$, i.e., $\xi \in \operatorname{lker} A_1$. Let $\bar{\xi}$ be such that $\bar{\xi} A_1 = b_1$. Then, $(\bar{\xi} + \alpha\xi)A_1 = b_1$ for any $\alpha \in \mathbb{R}$. Since $\mathcal{X}_1 \subseteq \mathcal{X}_2$, then we also have $(\bar{\xi} + \alpha\xi)A_2 = b_2$. This leads to

$$\frac{\bar{\xi}}{\alpha} A_2 + \xi A_2 = \frac{b_2}{\alpha}.$$

For $\alpha \to \infty$, this implies that $\xi A_2 = 0$. As such we have

$$\begin{bmatrix} \xi & 0 \end{bmatrix} \in \operatorname{lker} \begin{bmatrix} A_2 \\ b_2 \end{bmatrix}.$$

In both cases, we have that

$$\begin{bmatrix} \xi & \eta \end{bmatrix} \in \operatorname{lker} \begin{bmatrix} A_2 \\ b_2 \end{bmatrix}.$$

Therefore, we can conclude that

$$\operatorname{lker} \begin{bmatrix} A_1 \\ b_1 \end{bmatrix} \subseteq \operatorname{lker} \begin{bmatrix} A_2 \\ b_2 \end{bmatrix}$$

which is equivalent to (14.13). □

Lemma 14.8. *Let $A \in \mathbb{C}^{k \times k}$ and $a \in \mathbb{C}^k$. Then, the following statements are equivalent:*

(a) *If* $\begin{bmatrix} A & a \end{bmatrix} \begin{bmatrix} \xi_1 \\ \eta_1 \end{bmatrix} = \begin{bmatrix} A & a \end{bmatrix} \begin{bmatrix} \xi_2 \\ \eta_2 \end{bmatrix}$, *then* $\eta_1 = \eta_2$.

(b) $\operatorname{rank} \begin{bmatrix} A & a \end{bmatrix} = \operatorname{rank} A + 1$.

Proof. Note that (a) holds if and only if $\ker \begin{bmatrix} A & a \end{bmatrix} = \ker A \times \{0\}$. It then follows from the rank-nullity theorem that the two statements are equivalent. □

Proof of Theorem 14.6. Let Σ_{M_0} denote the set of all $\begin{bmatrix} q & -p \end{bmatrix}$ satisfying (14.9). Then, by Definition 14.5, the data (U, Y) are informative for interpolation at μ if and only if there exists a unique M_0 such that

$$\Sigma_{(U,Y)} \subseteq \Sigma_{M_0}. \tag{14.15}$$

We will show that the condition (14.11) is equivalent to the existence of M_0 satisfying (14.15) whereas the condition (14.12) is equivalent to the uniqueness of M_0.

First, note that $\Sigma_{(U,Y)}$ contains the true system and hence is nonempty. Therefore, it follows from Lemma 14.7 that there exists an M_0 satisfying (14.11) if and only if

$$\begin{bmatrix} \gamma_n(\mu) \\ M_0\gamma_n(\mu) \end{bmatrix} \in \operatorname{im} \begin{bmatrix} H_{n+1}(u_{[0,T-1]}) \\ H_{n+1}(y_{[0,T-1]}) \end{bmatrix}$$

or equivalently the existence of $\xi \in \mathbb{C}^{T-n}$ such that

$$\begin{bmatrix} H_{n+1}(u_{[0,T-1]}) & 0 \\ H_{n+1}(y_{[0,T-1]}) & -\gamma_n(\mu) \end{bmatrix} \begin{bmatrix} \xi \\ M_0 \end{bmatrix} = \begin{bmatrix} \gamma_n(\mu) \\ 0 \end{bmatrix}. \tag{14.16}$$

Consequently, we see that the existence of an M_0 satisfying (14.15) is equivalent to the condition (14.11).

Now, note that the uniqueness of M_0 is equivalent to the implication

$$\begin{bmatrix} H_{n+1}(u_{[0,T-1]}) & 0 \\ H_{n+1}(y_{[0,T-1]}) & -\gamma_n(\mu) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \eta_1 \end{bmatrix} = \begin{bmatrix} H_{n+1}(u_{[0,T-1]}) & 0 \\ H_{n+1}(y_{[0,T-1]}) & -\gamma_n(\mu) \end{bmatrix} \begin{bmatrix} \xi_2 \\ \eta_2 \end{bmatrix} \implies \eta_1 = \eta_2.$$

As such, Lemma 14.8 implies that the uniqueness of M_0 is equivalent to (14.12). ■

It readily follows from Theorem 14.6 that data informativity for interpolation at $\mu \in \mathbb{C} \setminus \mathbb{R}$ is equivalent to informativity at its complex conjugate.

An important consequence of Theorem 14.6 is that the data do not need to be informative for system identification in order to be for interpolation. Thus, it is possible that infinitely many systems are consistent with the data and they all have the same moment at a given interpolation point.

14.3.1 Illustrative example

We illustrate Theorem 14.6 by the following example.

Example 14.9. Consider the RL circuit depicted in Figure 14.1, which is a slight extension of [84, Example 22]. We take the currents through the inductors

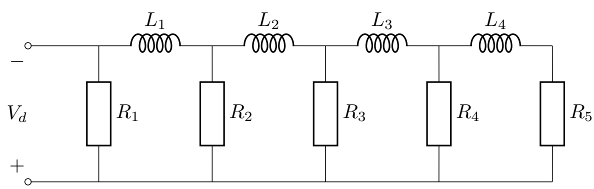


Figure 14.1: RL circuit with four inductors and five resistors.

L_1, L_2, L_3 and L_4 as the states of the system, so $n = 4$. The input is the voltage

V_d. Finally, as the output, we take the current through the first inductor L_1. This leads to the continuous-time dynamical system

$$\dot{x} = \begin{bmatrix} -\frac{R_2}{L_1} & \frac{R_2}{L_1} & 0 & 0 \\ \frac{R_2}{L_2} & \frac{-(R_2+R_3)}{L_2} & \frac{R_3}{L_2} & 0 \\ 0 & \frac{R_3}{L_3} & \frac{-(R_3+R_4)}{L_3} & \frac{R_4}{L_3} \\ 0 & 0 & \frac{R_4}{L_4} & \frac{-(R_4+R_5)}{L_4} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x. \tag{14.17}$$

Let $L_1 = L_2 = L_3 = L_4 = 1\ H$, $R_1 = 0.5\ \Omega$, $R_2 = 8\ \Omega$, $R_3 = 5\ \Omega$, $R_4 = 1\ \Omega$, and $R_5 = 4\ \Omega$. Consider the zero-order hold discretization (with the sampling period 0.2 s) of the system (14.17) which is assumed to be unknown. We apply the true system the input $U_{[0,20]}$ generated by an autonomous discrete-time system of the form

$$u(t) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} w(t) \quad \text{and} \quad w(t+1) = \begin{bmatrix} \sqrt{2} & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} w(t)$$

with $w(0) = \begin{bmatrix} 1 & 0 & 0 & -0.1 \end{bmatrix}^\top$ and harvest the output $Y_{[0,20]}$. This leads to the samples depicted in Figure 14.2.

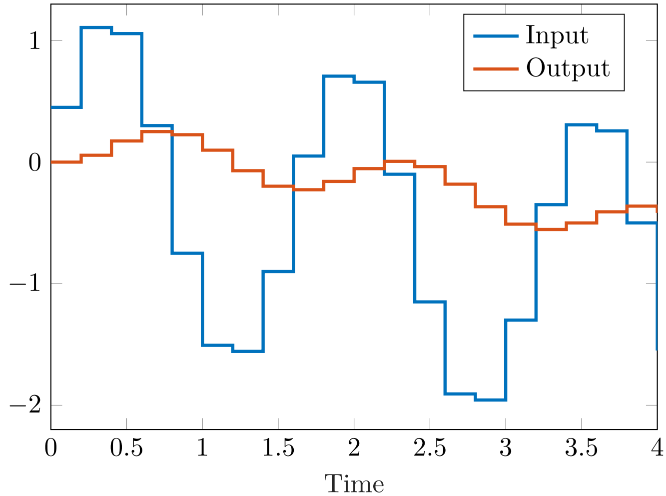


Figure 14.2: Input-output data with sampling period $\Delta = 0.2$ s.

It can be verified that the condition (14.5) does not hold for these data. As such, the data are not informative for system identification. Instead, there are (infinitely) many systems of the form (14.2) with order $n = 4$ that are consistent with the data.

Suppose that we aim at interpolation at $\mu_1 = 1$, $\mu_{2,3} = \frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$, and $\mu_{4,5} = \pm i$. One can verify by Theorem 14.6 that the data (U, Y) are informative for interpolation at μ_1, μ_2 and μ_3. Similarly, it can be verified that the data are *not* informative for interpolation at $\mu_{4,5}$. Finally, the moments of order 0 at μ_1, μ_2 and μ_3 are given by 1.575, $0.0031 - 0.1417i$ and $0.0031 + 0.1417i$, respectively, which are obtained by solving (14.16). ■

14.4 Higher order moments

In this section, we deal with higher order moments. For systems of the form (14.2), higher order moments can be defined in a recursive manner as follows.

Definition 14.10 (k-th moment)**.** Let $k \geqslant 1$. Given an interpolation point $\mu \in \mathbb{C}$ and j-th moments at μ, M_j for $j \in [0, k-1]$, a scalar $M_k \in \mathbb{C}$ is said to be a *k-th moment at μ* of the system (14.2) if

$$Q^{(k)}(\mu) = \sum_{j=0}^{k} \binom{k}{j} M_j P^{(k-j)}(\mu) \tag{14.18}$$

where

$$\binom{k}{j} = \frac{k!}{j!(k-j)!}$$

is the binomial coefficient and $f^{(j)}$ denotes the j-th derivative of f.

By using the γ notation defined in (14.10), we can rewrite (14.18) as a linear equation in $\begin{bmatrix} q & -p \end{bmatrix}$:

$$\begin{bmatrix} q & -p \end{bmatrix} \begin{bmatrix} \gamma_n^{(k)}(\mu) \\ \sum_{j=0}^{k} \binom{k}{j} M_j \gamma_{n-1}^{(k-j)}(\mu) \end{bmatrix} = \sum_{j=0}^{k} \binom{k}{j} \frac{n!}{(n-k+j)!} M_j \mu^{n-k+j} \tag{14.19}$$

where $\gamma_\ell^{(j)}$ denotes the j-th derivative of γ_ℓ.

Informativity of the data for higher order moment matching can also be defined in a recursive manner where moment matching of order 0 is understood as interpolation.

Definition 14.11. The data (U, Y) are *informative for moment matching of order k at μ* if

(a) the data are informative for moment matching of order j at μ for $j \in [0, k-1]$, and

(b) there exists a unique M_k such that (14.19) holds for all $\begin{bmatrix} q & -p \end{bmatrix} \in \Sigma_{(U,Y)}$.

By employing Lemmas 14.7 and 14.8 and following the footsteps of the proof of Theorem 14.6, one can prove the following necessary and sufficient conditions for informativity for higher order moment matching.

Theorem 14.12. *Let $k \geqslant 1$. Suppose that the data (U, Y) are informative for moment matching of order j at μ for all $j \in [0, k-1]$. Let M_j denote the corresponding moments. Then, the data (U, Y) are informative for moment matching of order k at μ if and only if*

$$\operatorname{rank} \begin{bmatrix} H_{n+1}(u_{[0,T-1]}) & 0 & \gamma_n^{(k)}(\mu) \\ H_{n+1}(y_{[0,T-1]}) & \gamma_n(\mu) & \sum_{j=0}^{k-1} \binom{k}{j} M_j \gamma_n^{(k-j)}(\mu) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} H_{n+1}(u_{[0,T-1]}) & 0 \\ H_{n+1}(y_{[0,T-1]}) & \gamma_n(\mu) \end{bmatrix}. \tag{14.20}$$

Next, we illustrate this result by means of an example.

Example 14.13. Consider the system and input-output data studied in Example 14.9. It can be checked that for $\sigma = 1$, condition (14.12) holds for $k = 1$. Hence, the data are informative for moment matching of order 1 at $\sigma = 1$. By solving the linear equation (14.19) (for $k = 1$) and using $M_0 = 1.575$, we obtain $M_1 = -31.8437$. ■

14.5 Multiple interpolation points

So far, our discussion considered a single interpolation point μ and its desired order of moment k. Let s pairs of interpolation points and their desired order of moments

$$\mathbb{P} = \{ (\mu_i, k_i) \mid i \in [1, s] \} \tag{14.21}$$

be given. We assume that $(\bar{\mu}_i, k_i) \in \mathbb{P}$ whenever $(\mu_i, k_i) \in \mathbb{P}$. By applying Theorems 14.6 and 14.12, one can verify whether the data are informative for moment matching for each pair. If so, (14.16) and (14.19) result in the corresponding moments

$$\mathbb{M}_i = \left\{ (\mu_i, M_j^i) \;\middle|\; j \in [0, k_i] \right\} \tag{14.22}$$

where M_j^i denotes the j-th moment at μ_i.

14.6 Reduced-order models by moment matching

In this section, we will investigate how reduced-order models can be computed from data that are informative for moment matching. As the results of Theorems 14.6 and 14.12 lead to the computation of moments at given interpolation points, obtaining such *a* reduced-order model is essentially a rational interpolation problem.

Let a reduced-order model of order r be given by

$$\hat{P}(\sigma)y = \hat{Q}(\sigma)u \tag{14.23}$$

where

$$\hat{P}(\xi) = \xi^r + \hat{p}_{r-1}\xi^{r-1} + \cdots + \hat{p}_1\xi + \hat{p}_0, \tag{14.24a}$$

$$\hat{Q}(\xi) = \hat{q}_r\xi^r + \hat{q}_{r-1}\xi^{r-1} + \cdots + \hat{q}_1\xi + \hat{q}_0. \tag{14.24b}$$

As before, we collect the parameters of (14.24) in vectors

$$\begin{aligned}\hat{p} &= \begin{bmatrix}\hat{p}_0 & \hat{p}_1 & \cdots & \hat{p}_{r-1}\end{bmatrix}\\ \hat{q} &= \begin{bmatrix}\hat{q}_0 & \hat{q}_1 & \cdots & \hat{q}_r\end{bmatrix}.\end{aligned}$$

Then, the model (14.23) interpolates or matches the 0-th moment at μ if M_0 in (14.16) satisfies $\hat{Q}(\mu) = M_0\hat{P}(\mu)$ which is equivalent to

$$\begin{bmatrix}\hat{q} & -\hat{p}\end{bmatrix}\begin{bmatrix}\gamma_r(\mu)\\ M_0\gamma_{r-1}(\mu)\end{bmatrix} = M_0\mu^r.$$

More generally, for $\mathbb{P}$ and $\mathbb{M}_i$ as in (14.21) and (14.22), a reduced-order model parameterized by $\begin{bmatrix}\hat{q} & -\hat{p}\end{bmatrix}$ must satisfy the linear equations

$$\begin{bmatrix}\hat{q} & -\hat{p}\end{bmatrix}\begin{bmatrix}\Gamma^r(\mu_i)\\ \Gamma^r_M(\mu_i)\end{bmatrix}_{1:(2r+1)} = \begin{bmatrix}\Gamma^r(\mu_i)\\ \Gamma^r_M(\mu_i)\end{bmatrix}_{(2r+2)} \tag{14.25}$$

for $i \in [1, s]$ where

$$\left[\begin{array}{c}\Gamma^r(\mu_i)\\ \hdashline \Gamma^r_M(\mu_i)\end{array}\right] = \left[\begin{array}{cccc}\gamma_r(\mu_i) & \gamma_r^{(1)}(\mu_i) & \cdots & \gamma_r^{(k_i)}(\mu_i)\\ \hdashline M_0^i\gamma_r(\mu_i) & M_0^i\gamma_r^{(1)}(\mu_i) + M_1^i\gamma_r(\mu_i) & \cdots & \sum_{j=0}^{k_i}\binom{k_i}{j}M_j^i\gamma_r^{(k_i-j)}(\mu_i)\end{array}\right].$$

Here, the notation $X_{1:(2r+1)}$ and $X_{(2r+2)}$ denote the first $2r+1$ rows and $(2r+2)$-th row of a matrix X, respectively.

Note that since $\begin{bmatrix}\hat{q} & -\hat{p}\end{bmatrix}$ is restricted to be real, then if $\mu_i \in \mathbb{C} \setminus \mathbb{R}$, we split (14.25) into its real and imaginary parts.

Now, we define

$$\hat{\Sigma}_{r,\mathbb{P}} = \left\{ \begin{bmatrix} \hat{q} & -\hat{p} \end{bmatrix} \in \mathbb{R}^{1\times(2r+1)} \mid (14.25) \text{ holds} \right\}.$$

Clearly, $\hat{\Sigma}_{r,\mathbb{P}}$ consists of all models of order r matching the moments M_j^i at the interpolation point σ_i for all $i \in [1, s]$ and $j \in [0, k_i]$. The following theorem readily follows from the solvability conditions of the linear equation given in (14.25).

Theorem 14.14. *Given the data of interpolation points $\mathbb{P}$ and moments $\mathbb{M}_i$ for $i \in [1, s]$ as in (14.21) and (14.22). Then, $\hat{\Sigma}_{r,\mathbb{P}} \neq \varnothing$ if and only if*

$$\operatorname{rank} \begin{bmatrix} \Gamma^r(\mu_1) & \Gamma^r(\mu_2) & \cdots & \Gamma^r(\mu_s) \\ \Gamma^r_M(\mu_1) & \Gamma^r_M(\mu_2) & \cdots & \Gamma^r_M(\mu_s) \end{bmatrix}_{1:(2r+1)}$$
$$=$$
$$\operatorname{rank} \begin{bmatrix} \Gamma^r(\mu_1) & \Gamma^r(\mu_2) & \cdots & \Gamma^r(\mu_s) \\ \Gamma^r_M(\mu_1) & \Gamma^r_M(\mu_2) & \cdots & \Gamma^r_M(\mu_s) \end{bmatrix}.$$

Note that we do not restrict $\begin{bmatrix} \hat{q} & -\hat{p} \end{bmatrix} \in \hat{\Sigma}_{r,\mathbb{P}}$ to be minimal, i.e., the polynomials $\hat{P}(\xi)$ and $\hat{Q}(\xi)$ might not be coprime. In addition, it is also obvious that if $r \geqslant k^* - 1$ where $k^* = \sum_{i=1}^{s}(k_i + 1)$, then (14.14) always holds.

The following example illustrates the computation of the reduced-order models.

Example 14.15. From Examples 14.9 and 14.13, we have

$$\mathbb{P} = \{(\mu_1, 1), (\mu_2, 0), (\mu_3, 0)\}$$

and $\mathbb{M}_i$ as follows:

$$\begin{aligned}
\mathbb{M}_1 &= \left\{(\mu_1, M_0^1), (\sigma_1, M_1^1)\right\} = \{(1, 1.575), (1, -31.8437)\}, \\
\mathbb{M}_2 &= \left\{(\mu_2, M_0^2)\right\} = \left\{\left(1/\sqrt{2} + i/\sqrt{2}, 0.0031 - 0.1417i\right)\right\}, \\
\mathbb{M}_3 &= \left\{(\mu_3, M_0^3)\right\} = \left\{\left(1/\sqrt{2} - i/\sqrt{2}, 0.0031 + 0.1417i\right)\right\}.
\end{aligned}$$

It can be checked that condition (14.14) does not hold for $r = 1$. Therefore, $\hat{\Sigma}_{1,\mathbb{P}} = \varnothing$. Meanwhile, it is satisfied with $r = 2$. Hence, $\hat{\Sigma}_{2,\mathbb{P}} \neq \varnothing$. We recall that (14.25) characterizes all reduced-order systems that achieve moment matching (i.e., the set $\hat{\Sigma}_{2,\mathbb{P}}$), of which two examples, in the form 14.23, are given by

$$(\sigma^2 - 1.191\sigma + 0.2268)y = (0.0001953\sigma^2 + 0.1236\sigma - 0.06692)u \tag{14.26}$$

and

$$(\sigma^2 - 1.911\sigma + 0.9116)y = (0.04622\sigma^2 + 0.02116\sigma - 0.06605)u. \tag{14.27}$$

The Bode plot of the reduced-order models (14.26) and (14.27) compared to the higher-order model of Example 14.9 are given in Figure 14.3 from which one can observe that the reduced-order model (14.26) captures the magnitude and phase of the higher-order model well, but (14.27) does not. Nevertheless, all of the curves are intersected at frequency 0 rad/s and $\frac{5\pi}{4}$ rad/s. This indicates that indeed these two systems in $\hat{\Sigma}_{2,\mathbb{P}}$ achieve the desired moment matching, but the optimality on approximating the higher-order model is another issue. We stress that this is a well-known feature of moment matching methods and not specific to our data-driven approach. ■

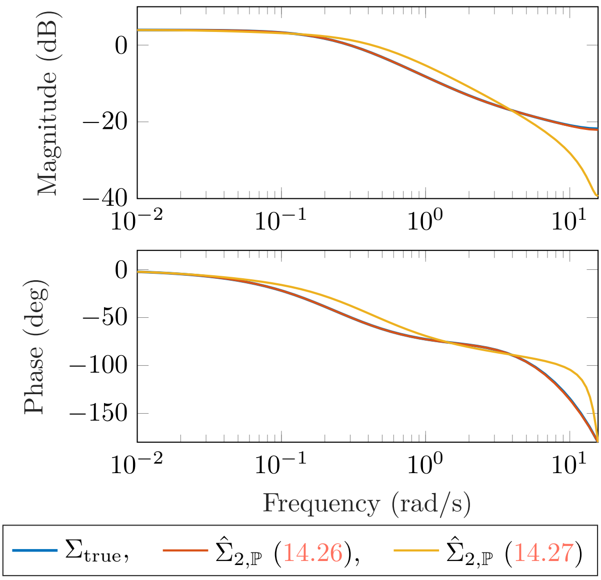


Figure 14.3: Comparison of the Bode plot of a second-order to that of the higher-order model.

Often reduced-order models are expected to preserve certain properties of the original models. Stability is one of the most common as well important property to be preserved. Next, we investigate conditions under which one can choose a stable reduced-order model. Note that stability of a reduced-order model $\begin{bmatrix}\hat{q} & -\hat{p}\end{bmatrix} \in \hat{\Sigma}_{r,\mathbb{P}}$ is purely determined by $\hat{p}$, i.e., one can choose $\hat{p}$ such that the roots of its corresponding polynomial are in the unit disc. Motivated by this observation, we provide a sufficient condition such that $\hat{p}$ can be chosen arbitrarily while $\begin{bmatrix}\hat{q} & -\hat{p}\end{bmatrix} \in \hat{\Sigma}_{r,\mathbb{P}}$.

Theorem 14.16. *For $\mathbb{P}$ and $\mathbb{M}_i$ as in (14.21) and (14.22), let $k^* = \sum_{i=1}^{s}(k_i+1)$.*

If $r \geqslant k^* - 1$ then for every $\hat{p} \in \mathbb{R}^{1\times r}$ there exists $\hat{q} \in \mathbb{R}^{1\times(r+1)}$ such that $\begin{bmatrix}\hat{q} & -\hat{p}\end{bmatrix} \in \hat{\Sigma}_{r,\mathbb{P}}$.

Proof. We know that $\begin{bmatrix}\hat{q} & -\hat{p}\end{bmatrix} \in \hat{\Sigma}_{r,\mathbb{P}}$ if and only if

$$\begin{bmatrix}\hat{q} & -\hat{p} & -1\end{bmatrix}\begin{bmatrix}\Gamma_1 \\ \Gamma_2\end{bmatrix} = 0, \tag{14.28}$$

where

$$\Gamma_1 = \begin{bmatrix}\Gamma^r(\mu_1) & \Gamma^r(\mu_2) & \cdots & \Gamma^r(\mu_s)\end{bmatrix}$$

and

$$\Gamma_2 = \begin{bmatrix}\Gamma^r_M(\mu_1) & \Gamma^r_M(\mu_2) & \cdots & \Gamma^r_M(\mu_s)\end{bmatrix}.$$

It is clear that $r \geqslant k^* - 1$ guarantees $\hat{\Sigma}_{r,\mathbb{P}} \neq \varnothing$. Particularly, since the structure of Γ_1 is a Vandermonde matrix, then $\operatorname{rank}(\Gamma_1) = k^*$ which implies (14.14). From (14.28), we see that for every $\hat{p}$ there exists $\hat{q}$ such that $\begin{bmatrix}\hat{q} & -\hat{p}\end{bmatrix} \in \hat{\Sigma}_{r,\mathbb{P}}$ if and only if

$$\operatorname{rsp}\Gamma_2 \subseteq \operatorname{rsp}\Gamma_1. \tag{14.29}$$

Since $\operatorname{rank}(\Gamma_1) = k^*$, we have that $\operatorname{rsp}\Gamma_1 = \mathbb{R}^{1\times k^*}$. Then, (14.29) readily holds. □

We close this chapter with an example that illustrates Theorem 14.16.

Example 14.17. Consider $\mathbb{P}$ as in Example 14.15. Let $r = 3$. Then, (14.14) readily holds. We desire to place the poles at $\{0.25, 0.4, 0.95\}$ to guarantee stability of the resulting reduced-order model. This corresponds to the choice $\hat{p} = \begin{bmatrix}-0.095 & 0.7175 & -1.6\end{bmatrix}$. By solving (14.25) with given $\hat{p}$, we obtain $\hat{q}$ that leads to the following reduced-order model:

$$(\sigma^3 - 1.6\sigma^2 + 0.7175\sigma - 0.095)y = (-0.05625\sigma^3 + 0.2624\sigma^2 - 0.2574\sigma + 0.08674)u.$$

■

14.7 Notes and references

A data-driven model reduction approach by moment matching has been introduced in [10, Chapter 4]. Relying on *frequency-domain* data, this so-called Loewner framework has strong connections to classical rational interpolation, see e.g. [6, 9, 11]. The Loewner framework (see e.g. , see [8, 108]) allows for obtaining reduced-order systems that achieve interpolation as well as further properties such as the preservation of stability [65], passivity, and optimal approximation in the $\mathcal{H}_2$ system norm [16]. To enable the use of time-domain data (rather than frequency-domain data) in this framework, [126] estimates transfer function values at given interpolation points by exploiting the relation between time- and frequency-domain data via the (discrete) Fourier transform, after which standard interpolatory methods can be used.

Data-driven model reduction from given *time-domain* data has also become an attractive topic in recent years. Belonging to the class of moment matching methods, algorithms for computing (a least-square approximation of) moments of linear or nonlinear systems are proposed in [141] and [123, Sec. VIA], building on the framework of [13]. These (estimated) moments are then used to construct families of reduced-order models. This method however relies on specifically chosen input data to guarantee that the resulting data (obtained from a steady-state response) is suitable for estimating a moment.

None of the existing works, however, investigate informativity of the data but often implicitly assumes it. The first paper that has introduced and characterized data informativity for moment matching is [30] on which this chapter is based. In this chapter, we focus on the theoretical aspects and do not discuss numerical applicability issues. An in-depth discussion on such issues can be found in [2].

Part V

Appendix

A

Mathematical background

This chapter deals with the mathematical material and basic notation. In Section A.1 we introduce basic mathematical notation and concepts. Also, some useful matrix theoretical lemmas are discussed. In Section A.2 we introduce quadratic matrix inequalities and discuss their solution sets. Section A.3 is devoted to an extensive treatment of matrix versions of Finsler's lemma and Yakubovich's S-lemma. The results of that section will be crucial in our treatment of data driven analysis and control in the context of noisy data.

For proofs of some of the results in this chapter we refer to the relevant references as stated in Section A.4 containing notes and references.

A.1 Basic notation, concepts and facts

We denote by $\mathbb{Z}_+$ the set of nonnegative integers and by $\mathbb{N}$ the set of positive integers. The set of real (respectively, complex) numbers is denoted by $\mathbb{R}$ (respectively, $\mathbb{C}$). We denote by $\operatorname{Re} c$ and $\operatorname{Im} c$ the real and imaginary parts of a complex number $c \in \mathbb{C}$. We will use the same notation to denote the real and imaginary parts of a row or column vector of complex numbers. The n-dimensional real Euclidean space is denoted by $\mathbb{R}^n$, while $\mathbb{C}^n$ denotes the space of n-tuples of complex numbers. The transpose of a vector $v \in \mathbb{C}^n$ is given by $v^\top$ and the complex conjugate transpose by v^*. The Euclidean norm of v is defined as $\|v\| := \sqrt{v^* v}$.

The space of real $m \times n$ matrices is denoted by $\mathbb{R}^{m\times n}$, and the space of complex $m \times n$ matrices by $\mathbb{C}^{m\times n}$. Some conventions on the notation that will be used for the zero matrix and identity matrix are as follows. If the dimensions are clear from the context we denote the zero matrix simply by 0 and the identity matrix by I. For given positive integers m and n we denote the $m\times n$ zero matrix by $0_{m\times n}$. The square $m \times m$ zero matrix is simply denoted by 0_m. Finally, the $m\times m$ identity matrix will be written as I_m. We use the notation $\|M\|$ to denote the spectral (or induced 2-norm) of a matrix $M \in \mathbb{C}^{m\times n}$, that is,

$$\|M\| := \sup \left\{ \frac{\|Mx\|}{\|x\|} \mid x \in \mathbb{C}^n, x \neq 0 \right\}.$$

In addition, $\operatorname{tr}(M)$ denotes the trace of $M \in \mathbb{R}^{m\times m}$.

For a matrix $M \in \mathbb{R}^{m\times n}$, we define its *image* by $\operatorname{im} M := \{Mx \mid x \in \mathbb{R}^n\}$, *kernel* by $\ker M := \{x \in \mathbb{R}^n \mid Mx = 0\}$, *row space* by $\operatorname{rsp} M := \{xM \mid x \in \mathbb{R}^{1\times m}\}$, and *left kernel* by $\operatorname{lker} M := \{x \in \mathbb{R}^{1\times m} \mid xM = 0\}$.

The set of eigenvalues of a given $M \in \mathbb{R}^{n\times n}$, called the spectrum of M, is denoted by $\sigma(M)$. We say that M is Schur if all its eigenvalues have modulus strictly less than 1. This is equivalent to the asymptotic stability of the associated discrete-time linear system $x(t+1) = Mx(t)$, where $x(t) \in \mathbb{R}^n$. Since in this book we focus solely on asymptotic stability, we simply say that M is *stable* if it is Schur.

Given a positive integer n, the subset of $\mathbb{R}^{n\times n}$ of all symmetric matrices will be denoted by $\mathbb{S}^n$. Symmetric matrices have only real eigenvalues. We denote by $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ the smallest and largest eigenvalue of a symmetric matrix M, respectively. For a given $M \in \mathbb{S}^n$, its number (counting multiplicities) of negative eigenvalues is denoted by $\operatorname{In}_-(M)$ and its number of positive eigenvalues by $\operatorname{In}_+(M)$. These numbers are called the negative and positive signature of M, respectively. In this context, the algebraic multiplicity of the zero eigenvalue of M is denoted by $\operatorname{In}_0(M)$. The triple $(\operatorname{In}_-(M), \operatorname{In}_0(M), \operatorname{In}_+(M))$ is called the inertia of M and is denoted by $\operatorname{In}(M)$.

If a matrix M is symmetric and positive semidefinite we will denote this simply by $M \geqslant 0$. Similarly, $M > 0$ will mean that M is symmetric and positive definite. Also, $M \geqslant N$ and $M > N$ will mean that M, N are symmetric matrices and $M - N$, respectively, positive semidefinite and positive definite. Likewise, we use the notation $M \leqslant 0$ and $M < 0$ to denote negative semidefiniteness and negative definiteness, respectively. We denote the unique positive semidefinite square root of a matrix $M \geqslant 0$ by $M^{\frac{1}{2}}$.

The dimension of a vector space $\mathcal{V}$ is denoted by $\dim \mathcal{V}$.

A.1.1 Generalized inverses

Given a real $m \times n$ matrix M, any matrix $M^\sharp$ with the property that

$$MM^\sharp M = M$$

is called a *generalized inverse* of M. If $\operatorname{rank} M = m$, i.e. M has full row rank, then every generalized inverse is a right-inverse, i.e. $MM^\sharp = I_m$. Conversely, every right-inverse of M is also a generalized inverse. Similarly, if $\operatorname{rank} M = n$, i.e., M has full column rank, then every generalized inverse is a left-inverse, i.e. $M^\sharp M = I_n$. Also, every left-inverse is a generalized inverse. Because of this, we will use the notation $M^\sharp$ both for generalized inverses as well as right-inverses (left-inverses) of M.

For any $m \times n$ matrix M there exists a generalized inverse with the additional properties that

$$
\begin{aligned}
&M^\sharp M M^\sharp = M^\sharp,\\
&M^\sharp M \text{ is symmetric},\\
&M M^\sharp \text{ is symmetric}.
\end{aligned}
$$

Indeed, since every matrix M can be factorized as $M = LR$ with L full column rank and R full row rank, a generalized inverse with these additional three properties is given by

$$M^\sharp := R^\top (RR^\top)^{-1}(L^\top L)^{-1}L^\top.$$

It can be shown that this particular generalized inverse is unique, in the sense that there is exactly one generalized inverse that satisfies the additional three properties. This particular generalized inverse is called the *Moore-Penrose pseudo-inverse* of M. It will be denoted in this book by $M^\dagger$.

The Moore-Penrose pseudo-inverse has several useful properties, among which we mention the following. For a given an $m \times n$ matrix M,

$$
\begin{aligned}
&(M^\dagger)^\dagger = M,\\
&I - M^\dagger M \text{ is the orthogonal projection of } \mathbb{R}^n \text{ onto } \ker M,\\
&(M^\top)^\dagger = (M^\dagger)^\top.
\end{aligned}
$$

In particular, the last property implies that the Moore-Penrose pseudo-inverse of a symmetric matrix M is again symmetric, i.e. $(M^\dagger)^\top = M^\dagger$.

A.1.2 Some relevant facts from matrix theory

In this section we formulate some useful lemmas on properties of matrices.

Lemma A.1. *Let $A \in \mathbb{R}^{r\times q}$ and $B \in \mathbb{R}^{p\times q}$.*

(a) *We have that $A^\top A \leqslant B^\top B$ if and only if there exists $S \in \mathbb{R}^{r\times p}$ such that*

$$A = SB \quad \text{and} \quad S^\top S \leqslant I. \tag{A.1}$$

(b) *Assume, in addition, that B has full column rank. Then $A^\top A < B^\top B$ if and only if there exists $S \in \mathbb{R}^{r\times p}$ such that*

$$A = SB \quad \text{and} \quad S^\top S < I. \tag{A.2}$$

Moreover, if $A^\top A - B^\top B \leqslant 0$ (respectively, < 0), then $S := AB^\dagger$ satisfies (A.1) *(respectively,* (A.2)*).*

Proof. The 'if' parts of statements (a) and (b) are straightforward. Indeed, assume that $A = SB$ with $S^\top S \leqslant I$ (respectively, $< I$). Then $A^\top A = B^\top S^\top SB \leqslant B^\top B$ (respectively, $< B^\top B$), where we have made use of full column rank of B to prove the strict inequality.

Next, we prove the 'only if' part of (a). We thus assume that $A^\top A \leqslant B^\top B$. Our goal is to show that $S := AB^\dagger$ satisfies (A.1). First, note that $A^\top A \leqslant B^\top B$ implies that $\ker B \subseteq \ker A$, equivalently, $\operatorname{im} A^\top \subseteq \operatorname{im} B^\top$. Thus, there exists a matrix $Z \in \mathbb{R}^{r\times q}$ such that $A^\top = B^\top Z^\top$, equivalently, $A = ZB$. Therefore, $SB = AB^\dagger B = ZBB^\dagger B = ZB = A$. Moreover, $S^\top S = (B^\dagger)^\top A^\top AB^\dagger \leqslant (B^\dagger)^\top B^\top BB^\dagger = BB^\dagger BB^\dagger = BB^\dagger \leqslant I$, where the last equality and the last inequality follow from the fact that $BB^\dagger$ is an orthogonal projection. This shows that S satisfies (A.1).

To prove the 'only if' part of statement (b), assume that B has full column rank and $A^\top A < B^\top B$. This implies that there exists an $\varepsilon > 0$ such that $(1+\varepsilon)A^\top A \leqslant B^\top B$. As such, by statement (a), the matrix $\bar{S} := \sqrt{1+\varepsilon}AB^\dagger$ satisfies $\sqrt{1+\varepsilon}A = \bar{S}B$ and $\bar{S}^\top \bar{S} \leqslant I$. Define $S := \frac{1}{\sqrt{1+\varepsilon}}\bar{S} = AB^\dagger$. Then $A = SB$ and $S^\top S = \frac{1}{1+\varepsilon}\bar{S}^\top \bar{S} < I$. We conclude that S satisfies (A.2) which proves the lemma. □

The following lemma is a direct consequence of [22, Prop. 6.1.7].

Lemma A.2. *Let $A \in \mathbb{R}^{q\times p}$ and $B \in \mathbb{R}^{r\times p}$. Then $AM = B$ if and only if $\operatorname{im} B \subseteq \operatorname{im} A$ and there exists $T \in \mathbb{R}^{r\times q}$ such that $M = A^\dagger B + (I_p - A^\dagger A)T$.*

A.2 Sets induced by quadratic matrix inequalities

In this book, an important role will be played by sets of matrices defined in terms of quadratic matrix inequalities (QMIs). We begin with sets of the form

$$\mathcal{Z}_r(\Pi) := \left\{ Z \in \mathbb{R}^{r\times q} \mid \begin{bmatrix} I_q \\ Z \end{bmatrix}^\top \Pi \begin{bmatrix} I_q \\ Z \end{bmatrix} \geqslant 0 \right\}, \tag{A.3}$$

where $\Pi \in \mathbb{S}^{q+r}$ is given. The very first question one may ask is: under what conditions on Π is the set $\mathcal{Z}_r(\Pi)$ nonempty? An immediate necessary condition is that Π must have at least q nonnegative eigenvalues. Clearly, this is not sufficient in general.

It follows from [22, Fact 8.15.28] that $Z \in \mathcal{Z}_r(\Pi)$ if and only if the matrix

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & -Z^\top \\ \Pi_{21} & \Pi_{22} & I \\ -Z & I & 0 \end{bmatrix} \tag{A.4}$$

has exactly r negative eigenvalues, where Π is partitioned as

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}. \tag{A.5}$$

From now on, whenever we partition a matrix $\Pi \in \mathbb{S}^{q+r}$ like (A.5), this means that Π_{11} is $q \times q$ and Π_{22} is $r \times r$. We denote by $\Pi \,|\, \Pi_{22} := \Pi_{11} - \Pi_{12}\Pi_{22}^{\dagger}\Pi_{21}$ the (generalized) Schur complement of Π with respect to Π_{22}. The condition on the eigenvalues of (A.4) does not translate to an easily verifiable condition on Π for nonemptiness of $\mathcal{Z}_r(\Pi)$. Nevertheless, it leads to a useful dualization result that will be crucial in some of the later chapters. To state this result, for given $\mathcal{S} \subseteq \mathbb{R}^{r\times q}$, we define $\mathcal{S}^\top := \left\{ Z^\top \mid Z \in \mathcal{S} \right\}$.

Lemma A.3. *Let $\Pi \in \mathbb{S}^{q+r}$ be such that $\operatorname{In}(\Pi) = (r, 0, q)$. Assume that $\mathcal{Z}_r(\Pi)$ is nonempty. Then, $\left(\mathcal{Z}_r(\Pi)\right)^\top = \mathcal{Z}_q(\Pi^\star_{r,q})$ where*

$$\Pi^\star_{r,q} := \begin{bmatrix} 0 & -I_r \\ I_q & 0 \end{bmatrix} \Pi^{-1} \begin{bmatrix} 0 & -I_q \\ I_r & 0 \end{bmatrix}.$$

In order to prove Lemma A.3, we need Haynsworth's inertia theorem, for which we refer to [22, Fact 6.5.5]. This result is recalled in the following lemma.

Lemma A.4. *Let $\Pi \in \mathbb{S}^{q+r}$. The following statements hold.*

- *If* $\ker \Pi_{22} \subseteq \ker \Pi_{12}$ *then* $\operatorname{In}(\Pi) = \operatorname{In}(\Pi_{22}) + \operatorname{In}(\Pi \,|\, \Pi_{22})$.
- *If* $\ker \Pi_{11} \subseteq \ker \Pi_{21}$ *then* $\operatorname{In}(\Pi) = \operatorname{In}(\Pi_{11}) + \operatorname{In}(\Pi \,|\, \Pi_{11})$.

Proof of Lemma A.3. Let

$$\begin{bmatrix} \hat{\Pi}_{11} & \hat{\Pi}_{12} \\ \hat{\Pi}_{12}^\top & \hat{\Pi}_{22} \end{bmatrix} := -\Pi^{-1}$$

where $\hat{\Pi}_{11} \in \mathbb{R}^{q\times q}$, $\hat{\Pi}_{12} \in \mathbb{R}^{q\times r}$, and $\hat{\Pi}_{22} \in \mathbb{R}^{r\times r}$. Also let $Z \in \mathbb{R}^{r\times q}$ and define

$$\Theta_Z := \begin{bmatrix} 0 & I & Z^\top \\ I & \hat{\Pi}_{11} & \hat{\Pi}_{12} \\ Z & \hat{\Pi}_{12}^\top & \hat{\Pi}_{22} \end{bmatrix}.$$

By Lemma A.4, we obtain

$$\operatorname{In}(\Theta_Z) = \operatorname{In}(-\Pi^{-1}) + \operatorname{In}\left(\begin{bmatrix} I \\ Z \end{bmatrix}^\top \Pi \begin{bmatrix} I \\ Z \end{bmatrix} \right). \tag{A.6}$$

Next, we define

$$N := \begin{bmatrix} 0 & I \\ I & \hat{\Pi}_{11} \end{bmatrix}.$$

Note that N is nonsingular and

$$N^{-1} = \begin{bmatrix} -\hat{\Pi}_{11} & I \\ I & 0 \end{bmatrix}.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_q$ be the eigenvalues of $\hat{\Pi}_{11}$. Denote the corresponding eigenvectors by $v_1, v_2, \ldots, v_q \in \mathbb{R}^q$. Then, it can be easily verified that for $i = 1, 2, \ldots, q$,

$$\mu_i^+ = \frac{\lambda_i + \sqrt{\lambda_i^2 + 4}}{2} \quad \text{and} \quad \mu_i^- = \frac{\lambda_i - \sqrt{\lambda_i^2 + 4}}{2}$$

are the $2q$ eigenvalues of N, with corresponding eigenvectors

$$w_i^+ = \begin{bmatrix} v_i \\ \mu_i^+ v_i \end{bmatrix} \quad \text{and} \quad w_i^- = \begin{bmatrix} v_i \\ \mu_i^- v_i \end{bmatrix} \quad \text{for } i = 1, 2, \ldots, q.$$

As such, N has precisely q positive and q negative eigenvalues. In other words, $\text{In}(N) = (q, 0, q)$. We also have that the Schur complement of Θ_Z with respect to N is given by

$$\begin{aligned}
\hat{\Pi}_{22} - \begin{bmatrix} Z & \hat{\Pi}_{12}^\top \end{bmatrix} \begin{bmatrix} -\hat{\Pi}_{11} & I \\ I & 0 \end{bmatrix} \begin{bmatrix} Z^\top \\ \hat{\Pi}_{12} \end{bmatrix} &= \hat{\Pi}_{22} + Z\hat{\Pi}_{11}Z^\top - Z\hat{\Pi}_{12} - \hat{\Pi}_{12}^\top Z^\top \\
&= \begin{bmatrix} I \\ Z^\top \end{bmatrix}^\top \begin{bmatrix} \hat{\Pi}_{22} & -\hat{\Pi}_{12}^\top \\ -\hat{\Pi}_{12} & \hat{\Pi}_{11} \end{bmatrix} \begin{bmatrix} I \\ Z^\top \end{bmatrix} \\
&= \begin{bmatrix} I \\ Z^\top \end{bmatrix}^\top \Pi_{r,q}^\star \begin{bmatrix} I \\ Z^\top \end{bmatrix}.
\end{aligned}$$

By Lemma A.4, this implies that

$$\text{In}(\Theta_Z) = \text{In}(N) + \text{In}\left(\begin{bmatrix} I \\ Z^\top \end{bmatrix}^\top \Pi_{r,q}^\star \begin{bmatrix} I \\ Z^\top \end{bmatrix} \right). \tag{A.7}$$

By combining (A.6) and (A.7) we obtain

$$\text{In}\left(\begin{bmatrix} I \\ Z \end{bmatrix}^\top \Pi \begin{bmatrix} I \\ Z \end{bmatrix} \right) = \text{In}\left(\begin{bmatrix} I \\ Z^\top \end{bmatrix}^\top \Pi_{r,q}^\star \begin{bmatrix} I \\ Z^\top \end{bmatrix} \right) + (0, 0, q - r)$$

since $\text{In}(-\Pi^{-1}) = (q, 0, r)$ and $\text{In}(N) = (q, 0, q)$. This implies that

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top \Pi \begin{bmatrix} I \\ Z \end{bmatrix} \geqslant 0 \iff \begin{bmatrix} I \\ Z^\top \end{bmatrix}^\top \Pi_{r,q}^\star \begin{bmatrix} I \\ Z^\top \end{bmatrix} \geqslant 0$$

which proves the lemma. □

It turns out that for particular matrices Π, a Schur complement argument on the matrix Π itself leads to a simple characterization of nonemptiness of the set $\mathcal{Z}_r(\Pi)$. Specifically, suppose that $\Pi_{22} \leqslant 0$ and $\ker \Pi_{22} \subseteq \ker \Pi_{12}$. Since the latter condition is equivalent to $\Pi_{12}\Pi_{22}\Pi_{22}^\dagger = \Pi_{12}$, we have that

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} = \begin{bmatrix} I_q & \Pi_{12}\Pi_{22}^\dagger \\ 0 & I_r \end{bmatrix} \begin{bmatrix} \Pi\,|\,\Pi_{22} & 0 \\ 0 & \Pi_{22} \end{bmatrix} \begin{bmatrix} I_q & 0 \\ \Pi_{22}^\dagger\Pi_{21} & I_r \end{bmatrix}. \tag{A.8}$$

This results in

$$\begin{bmatrix} I_q \\ Z \end{bmatrix}^\top \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \begin{bmatrix} I_q \\ Z \end{bmatrix} = \Pi\,|\,\Pi_{22} + (Z + \Pi_{22}^\dagger\Pi_{21})^\top \Pi_{22} (Z + \Pi_{22}^\dagger\Pi_{21}) \tag{A.9}$$

and, since $\Pi_{22} \leqslant 0$,

$$\Pi\,|\,\Pi_{22} = \begin{bmatrix} I_q \\ -\Pi_{22}^\dagger\Pi_{21} \end{bmatrix}^\top \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \begin{bmatrix} I_q \\ -\Pi_{22}^\dagger\Pi_{21} \end{bmatrix} \geqslant \begin{bmatrix} I_q \\ Z \end{bmatrix}^\top \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \begin{bmatrix} I_q \\ Z \end{bmatrix} \tag{A.10}$$

for any $Z \in \mathbb{R}^{r \times q}$. The conclusion is that if $\Pi_{22} \leqslant 0$ and $\ker \Pi_{22} \subseteq \ker \Pi_{12}$ then $\mathcal{Z}_r(\Pi)$ is nonempty if and only if $\Pi\,|\,\Pi_{22} \geqslant 0$. Motivated by this observation, we define the set

$$\boldsymbol{\Pi}_{q,r} = \left\{ \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \in \mathbb{S}^{q+r} \mid \Pi_{22} \leqslant 0, \Pi\,|\,\Pi_{22} \geqslant 0 \text{ and } \ker \Pi_{22} \subseteq \ker \Pi_{12} \right\}. \tag{A.11}$$

Note that it follows from the definition of $\boldsymbol{\Pi}_{q,r}$ that the set $\mathcal{Z}_r(\Pi)$ is nonempty for all $\Pi \in \boldsymbol{\Pi}_{q,r}$. Next, in the following subsection, for $\Pi \in \boldsymbol{\Pi}_{q,r}$ we will investigate basic properties of the sets $\mathcal{Z}_r(\Pi)$ and the following closely related sets

$$\mathcal{Z}_r^+(\Pi) := \left\{ Z \in \mathbb{R}^{r \times q} \mid \begin{bmatrix} I_q \\ Z \end{bmatrix}^\top \Pi \begin{bmatrix} I_q \\ Z \end{bmatrix} > 0 \right\} \tag{A.12}$$

$$\mathcal{Z}_r^0(\Pi) := \left\{ Z \in \mathbb{R}^{r \times q} \mid \begin{bmatrix} I_q \\ Z \end{bmatrix}^\top \Pi \begin{bmatrix} I_q \\ Z \end{bmatrix} = 0 \right\}. \tag{A.13}$$

A.2.1 Basic properties

In the following theorem, we study nonemptiness, convexity, and boundedness of the sets induced by QMIs as introduced above.

Theorem A.5. *Let $\Pi \in \boldsymbol{\Pi}_{q,r}$. Then, $\mathcal{Z}_r(\Pi)$*

(a) *is nonempty and convex.*

(b) *is bounded if and only if* $\Pi_{22} < 0$.

(c) *has nonempty interior if and only if* $\Pi_{22} = 0$ *or* $\Pi \,|\, \Pi_{22} > 0$.

Further,

(d) $\mathcal{Z}_r^+(\Pi)$ *is nonempty if and only if* $\Pi \,|\, \Pi_{22} > 0$.

(e) $\mathcal{Z}_r^0(\Pi)$ *is nonempty if and only if* $\operatorname{rank} \Pi_{22} \geqslant \operatorname{rank} \Pi \,|\, \Pi_{22}$.

Proof. (a): Since $\Pi \,|\, \Pi_{22} \geqslant 0$, it follows from (A.10) that $-\Pi_{22}^{\dagger}\Pi_{21} \in \mathcal{Z}_r(\Pi)$. This proves nonemptiness whereas convexity readily follows from $\Pi_{22} \leqslant 0$.

(b): We first prove the 'if' part. Let $Z \in \mathcal{Z}_r(\Pi)$. Then, it follows from (A.9) that $(Z+\Pi_{22}^{\dagger}\Pi_{21})^\top(-\Pi_{22})(Z+\Pi_{22}^{\dagger}\Pi_{21}) \leqslant \Pi \,|\, \Pi_{22}$. This leads to $\lambda_{\min}(-\Pi_{22})(Z+\Pi_{22}^{\dagger}\Pi_{21})^\top(Z + \Pi_{22}^{\dagger}\Pi_{21}) \leqslant \lambda_{\max}(\Pi \,|\, \Pi_{22})I$. Since $-\Pi_{22} > 0$ and $\Pi \,|\, \Pi_{22} \geqslant 0$, we see that $\|Z + \Pi_{22}^{\dagger}\Pi_{21}\| \leqslant \alpha$ for some $\alpha \geqslant 0$. Hence, $\mathcal{Z}_r(\Pi)$ is bounded.

For the 'only if' part, let $Z \in \mathcal{Z}_r(\Pi)$ and let $\xi \in \mathbb{R}^r$ be such that $\Pi_{22}\xi = 0$. Since $\Pi \in \mathbf{\Pi}_{q,r}$, we see that $Z + \alpha\xi\xi^\top \in \mathcal{Z}_r(\Pi)$ for any $\alpha \in \mathbb{R}$. Since $\mathcal{Z}_r(\Pi)$ is bounded, this implies $\xi = 0$. This proves that Π_{22} is nonsingular. Thus $\Pi_{22} \leqslant 0$ implies $\Pi_{22} < 0$.

(c): For the 'if' part, let $\Delta \in \mathbb{R}^{r\times q}$ be such that $\|\Delta\| \leqslant 1$, equivalently, $\Delta^\top\Delta \leqslant I$. For all $\varepsilon > 0$, we have

$$\begin{aligned}\Pi \,|\, \Pi_{22} + \varepsilon^2\Delta^\top\Pi_{22}\Delta &\geqslant \lambda_{\min}(\Pi \,|\, \Pi_{22})I + \varepsilon^2\lambda_{\min}(\Pi_{22})\Delta^\top\Delta \\ &\geqslant \lambda_{\min}(\Pi \,|\, \Pi_{22})I + \varepsilon^2\lambda_{\min}(\Pi_{22})I\end{aligned}$$

where the last inequality follows from the facts that $\Pi_{22} \leqslant 0$ and $\Delta^\top\Delta \leqslant I$. If $\Pi_{22} = 0$, then the right hand side is nonnegative for any ε since $\Pi \,|\, \Pi_{22} \geqslant 0$. If $\Pi \,|\, \Pi_{22} > 0$, then the right hand side is nonnegative for all sufficiently small $\varepsilon > 0$. Therefore, there exists $\varepsilon > 0$ such that

$$\Pi \,|\, \Pi_{22} + \varepsilon^2\Delta^\top\Pi_{22}\Delta \geqslant 0 \tag{A.14}$$

for all Δ with $\|\Delta\| \leqslant 1$. Now, take $Z_0 = -\Pi_{22}^{\dagger}\Pi_{21}$ and note that

$$\Pi \,|\, \Pi_{22} + (Z_0 + \varepsilon\Delta + \Pi_{22}^{\dagger}\Pi_{21})^\top\Pi_{22}(Z_0 + \varepsilon\Delta + \Pi_{22}^{\dagger}\Pi_{21}) = \Pi \,|\, \Pi_{22} + \varepsilon^2\Delta^\top\Pi_{22}\Delta \geqslant 0$$

for all Δ with $\|\Delta\| \leqslant 1$ due to (A.14). Then, it follows from (A.9) that $Z_0 + \varepsilon\Delta \in \mathcal{Z}_r(\Pi)$ for all Δ with $\|\Delta\| \leqslant 1$. This means that the set $\mathcal{Z}_r(\Pi)$ has nonempty interior.

For the 'only if' part, suppose that Z_0 is in the interior of $\mathcal{Z}_r(\Pi)$. This means that there exists $\varepsilon > 0$ such that $Z_0 + \varepsilon\Delta \in \mathcal{Z}_r(\Pi)$ for all Δ with $\|\Delta\| \leqslant 1$. By (A.9),

$$\Pi \,|\, \Pi_{22} + (Z_0 + \varepsilon\Delta + \Pi_{22}^{\dagger}\Pi_{21})^\top\Pi_{22}(Z_0 + \varepsilon\Delta + \Pi_{22}^{\dagger}\Pi_{21}) \geqslant 0. \tag{A.15}$$

Suppose that $\xi \in \mathbb{R}^q$ is such that $(\Pi \,|\, \Pi_{22})\xi = 0$. Since $\Pi_{22} \leqslant 0$, (A.15) yields the equation $\Pi_{22}(Z_0 + \varepsilon\Delta + \Pi_{22}^\dagger\Pi_{21})\xi = 0$ for all Δ with $\|\Delta\| \leqslant 1$. By taking $\Delta = 0$, we see that $\Pi_{22}(Z_0 + \Pi_{22}^\dagger\Pi_{21})\xi = 0$. Therefore, $\Pi_{22}\Delta\xi = 0$ for all Δ with $\|\Delta\| \leqslant 1$. In particular, consider $\Delta = \zeta\xi^\top$ where $\zeta \in \mathbb{R}^r$. Then, we conclude that $\Pi_{22}\zeta\xi^\top\xi = 0$ for all $\zeta \in \mathbb{R}^r$. Therefore, either $\Pi_{22} = 0$ or $\xi = 0$. Equivalently, either $\Pi_{22} = 0$ or $\Pi \,|\, \Pi_{22} > 0$.

(d): For the 'if' part, suppose that $\Pi \,|\, \Pi_{22} > 0$. Then, it follows from (A.10) that $-\Pi_{22}^\dagger\Pi_{21} \in \mathcal{Z}_r^+(\Pi)$. Thus, $\mathcal{Z}_r^+(\Pi)$ is nonempty. For the 'only if' part, suppose that $\mathcal{Z}_r^+(\Pi)$ is nonempty. Let $Z \in \mathcal{Z}_r^+(\Pi)$. Then, (A.10) implies that $\Pi \,|\, \Pi_{22} > 0$.

(e): For the 'only if' part, suppose that $\mathcal{Z}_r^0(\Pi)$ is nonempty. Let $Z \in \mathcal{Z}_r^0(\Pi)$. Then, it follows from (A.9) that $\Pi \,|\, \Pi_{22} = -(Z + \Pi_{22}^\dagger\Pi_{21})^\top\Pi_{22}(Z + \Pi_{22}^\dagger\Pi_{21})$. Since the rank of a product of matrices is less than or equal to the ranks of individual matrices, we see that $\operatorname{rank}\Pi_{22} \geqslant \operatorname{rank}(\Pi \,|\, \Pi_{22})$. For the 'if' part, suppose that $\operatorname{rank}\Pi_{22} \geqslant \operatorname{rank}(\Pi \,|\, \Pi_{22})$. Let $U_1\Sigma_1U_1^\top$ and $U_2\Sigma_2U_2^\top$ be eigenvalue decompositions of $\Pi \,|\, \Pi_{22}$ and $-\Pi_{22}$, respectively. Then, $\operatorname{rank}\Sigma_2 \geqslant \operatorname{rank}\Sigma_1$. Hence, there exists a diagonal matrix $D \geqslant 0$ such that $\Sigma_1 = D\Sigma_2$. Take $\bar{Z} = -\Pi_{22}^\dagger\Pi_{21} + U_2D^{\frac{1}{2}}U_1^T$. Note that $(\bar{Z} + \Pi_{22}^\dagger\Pi_{21})^\top\Pi_{22}(\bar{Z} + \Pi_{22}^\dagger\Pi_{21}) = -U_1\Sigma_1U_1^\top = -\Pi \,|\, \Pi_{22}$. Consequently, it follows from (A.9) that $\bar{Z} \in \mathcal{Z}_r^0(\Pi)$ and $\mathcal{Z}_r^0(\Pi)$ is nonempty. □

A.2.2 Parameterization of $\mathcal{Z}_r(\Pi)$ and $\mathcal{Z}_r^+(\Pi)$

It turns out that one can explicitly parameterize all solutions of a given QMI associated with $\Pi \in \mathbf{\Pi}_{q,r}$, as stated in the following theorem.

Theorem A.6. *Let $\Pi \in \mathbf{\Pi}_{q,r}$. The following statements hold:*

(a) *$Z \in \mathcal{Z}_r(\Pi)$ if and only if*

$$Z = -\Pi_{22}^\dagger\Pi_{21} + \big((-\Pi_{22})^\dagger\big)^{\frac{1}{2}} S(\Pi \,|\, \Pi_{22})^{\frac{1}{2}} + (I - \Pi_{22}^\dagger\Pi_{22})T \tag{A.16}$$

for some $S, T \in \mathbb{R}^{r\times q}$ with $S^\top S \leqslant I$.

(b) *Assume that $\mathcal{Z}_r^+(\Pi)$ is nonempty, equivalently, $\Pi \,|\, \Pi_{22} > 0$. Then, $Z \in \mathcal{Z}_r^+(\Pi)$ if and only if*

$$Z = -\Pi_{22}^\dagger\Pi_{21} + \big((-\Pi_{22})^\dagger\big)^{\frac{1}{2}} S(\Pi \,|\, \Pi_{22})^{\frac{1}{2}} + (I - \Pi_{22}^\dagger\Pi_{22})T \tag{A.17}$$

for some $S, T \in \mathbb{R}^{r\times q}$ with $S^\top S < I$.

Proof. We first prove (b). From (A.9) we have that $Z \in \mathcal{Z}_r^+(\Pi)$ if and only if

$$(Z + \Pi_{22}^\dagger\Pi_{21})^\top(-\Pi_{22})(Z + \Pi_{22}^\dagger\Pi_{21}) < \Pi \,|\, \Pi_{22}. \tag{A.18}$$

By Lemma A.1.(b), we then have that $Z \in \mathcal{Z}_r^+(\Pi)$ if and only if there exists a matrix S such that $S^\top S < I$ and $(-\Pi_{22})^{\frac{1}{2}}(Z + \Pi_{22}^\dagger \Pi_{21}) = S(\Pi \,|\, \Pi_{22})^{\frac{1}{2}}$. Using the fact that $\ker(-\Pi_{22})^{\frac{1}{2}} = \ker \Pi_{22}$, and by exploiting Lemma A.2, we see that this is equivalent to $Z + \Pi_{22}^\dagger \Pi_{21} = \big((-\Pi_{22})^{\frac{1}{2}}\big)^\dagger S(\Pi \,|\, \Pi_{22})^{\frac{1}{2}} + (I - \Pi_{22}^\dagger \Pi_{22})T$ for some $T \in \mathbb{R}^{r\times q}$. This proves (b). The proof of (a) follows the same arguments but instead invokes Lemma A.1.(a). □

A.2.3 Image of $\mathcal{Z}_r(\Pi)$ and $\mathcal{Z}_r^+(\Pi)$ under linear maps

Let $W \in \mathbb{R}^{q\times p}$. For $\mathcal{S} \subseteq \mathbb{R}^{r\times q}$, we define $\mathcal{S}W := \{SW \mid S \in \mathcal{S}\}$. Also, for $\Pi \in \mathbb{S}^{q+r}$ we define

$$\Pi_W := \begin{bmatrix} W^\top & 0 \\ 0 & I_r \end{bmatrix} \Pi \begin{bmatrix} W & 0 \\ 0 & I_r \end{bmatrix} = \begin{bmatrix} W^\top \Pi_{11} W & W^\top \Pi_{12} \\ \Pi_{21} W & \Pi_{22} \end{bmatrix} \in \mathbb{S}^{p+r}. \tag{A.19}$$

Note that

$$\Pi_W \in \mathbf{\Pi}_{p,r} \tag{A.20}$$

provided that $\Pi \in \mathbf{\Pi}_{q,r}$. Next, we will study the relationship between the sets $\mathcal{Z}_r(\Pi)$ and $\mathcal{Z}_r(\Pi_W)$.

Theorem A.7. *Let $\Pi \in \mathbf{\Pi}_{q,r}$ and $W \in \mathbb{R}^{q\times p}$. We have that $\mathcal{Z}_r(\Pi)W \subseteq \mathcal{Z}_r(\Pi_W)$. Assume, in addition, that at least one of the following two conditions hold:*

(a) *W has full column rank.*

(b) *Π_{22} is nonsingular.*

Then, $\mathcal{Z}_r(\Pi)W = \mathcal{Z}_r(\Pi_W)$.

Proof. First we prove that $\mathcal{Z}_r(\Pi)W \subseteq \mathcal{Z}_r(\Pi_W)$. Let $Z' \in \mathcal{Z}_r(\Pi)W$. Then, $Z' = ZW$ where $Z \in \mathcal{Z}_r(\Pi)$, that is

$$\begin{bmatrix} I_q \\ Z \end{bmatrix}^\top \Pi \begin{bmatrix} I_q \\ Z \end{bmatrix} \geqslant 0.$$

By pre- and post-multiplying by $W^\top$ and W, we obtain

$$W^\top \begin{bmatrix} I_q \\ Z \end{bmatrix}^\top \Pi \begin{bmatrix} I_q \\ Z \end{bmatrix} W = \begin{bmatrix} I_p \\ Z' \end{bmatrix}^\top \Pi_W \begin{bmatrix} I_p \\ Z' \end{bmatrix} \geqslant 0.$$

This means that $Z' \in \mathcal{Z}_r(\Pi_W)$ and hence $\mathcal{Z}_r(\Pi)W \subseteq \mathcal{Z}_r(\Pi_W)$.

Now, we assume that at least one of the conditions on W and Π_{22} hold. We claim that $\mathcal{Z}_r(\Pi_W) \subseteq \mathcal{Z}_r(\Pi)W$. Let $Z' \in \mathcal{Z}_r(\Pi_W)$. Note that $\Pi_W \,|\, \Pi_{22} = W^\top(\Pi \,|\, \Pi_{22})W$. From (A.19) and Theorem A.6.(a), we see that

$$Z' = -\Pi_{22}^\dagger \Pi_{21} W + \big((-\Pi_{22})^\dagger\big)^{\frac{1}{2}} S \big(W^\top(\Pi \,|\, \Pi_{22})W\big)^{\frac{1}{2}} + (I - \Pi_{22}^\dagger \Pi_{22})V \tag{A.21}$$

where $V, S \in \mathbb{R}^{r\times p}$ with $S^\top S \leqslant I_p$. Since $\big(W^\top(\Pi \,|\, \Pi_{22})W\big)^{\frac{1}{2}} \big(W^\top(\Pi \,|\, \Pi_{22})W\big)^{\frac{1}{2}} = W^\top(\Pi \,|\, \Pi_{22})^{\frac{1}{2}}(\Pi \,|\, \Pi_{22})^{\frac{1}{2}}W$, due to Lemma A.1 we have that $\big(W^\top(\Pi \,|\, \Pi_{22})W\big)^{\frac{1}{2}} = T(\Pi \,|\, \Pi_{22})^{\frac{1}{2}}W$ where $T \in \mathbb{R}^{p\times q}$ is such that $T^\top T \leqslant I_q$. If W has full column rank then (A.21) results in $Z' = ZW$ where

$$Z := -\Pi_{22}^\dagger \Pi_{21} + \big((-\Pi_{22})^\dagger\big)^{\frac{1}{2}} ST(\Pi \,|\, \Pi_{22})^{\frac{1}{2}} + (I - \Pi_{22}^\dagger \Pi_{22})V(W^\top W)^{-1}W^\top. \tag{A.22}$$

On the other hand, if Π_{22} is nonsingular then $I - \Pi_{22}^\dagger \Pi_{22} = 0$ and $Z' = ZW$ with

$$Z := -\Pi_{22}^{-1}\Pi_{21} + \big(-\Pi_{22}^{-1}\big)^{\frac{1}{2}} ST(\Pi \,|\, \Pi_{22})^{\frac{1}{2}}. \tag{A.23}$$

In either of these two cases, we observe that $T^\top S^\top ST \leqslant T^\top T \leqslant I_q$. Therefore, Theorem A.6.(a) implies that $Z \in \mathcal{Z}_r(\Pi)$. Consequently, we see that $Z' = ZW$ for some $Z \in \mathcal{Z}_r(\Pi)$ and thus $\mathcal{Z}_r(\Pi_W) \subseteq \mathcal{Z}_r(\Pi)W$. This proves the theorem. ☐

A similar result holds for the sets $\mathcal{Z}_r^+(\Pi)$ and $\mathcal{Z}_r^+(\Pi_W)$, as shown next.

Theorem A.8. *Let $\Pi \in \mathbf{\Pi}_{q,r}$ and $W \in \mathbb{R}^{q\times p}$. Assume that W has full column rank and $\mathcal{Z}_r^+(\Pi)$ is nonempty. Then, $\mathcal{Z}_r^+(\Pi)W = \mathcal{Z}_r^+(\Pi_W)$.*

The proof of Theorem A.8 is similar to that of Theorem A.7, but applies Theorem A.6.(b) instead of Theorem A.6.(a).

The following two corollaries follow from Theorems A.7 and A.8 and provide conditions under which there exists a 'structured' matrix in $\mathcal{Z}_r(\Pi)$ (respectively, $\mathcal{Z}_r^+(\Pi)$) that satisfies a linear equation.

Corollary A.9. *Let $\Pi \in \mathbb{S}^{q+r}$ with $\Pi_{22} \leqslant 0$ and $\ker \Pi_{22} \subseteq \ker \Pi_{21}$, $W \in \mathbb{R}^{q\times p}$, and $Y \in \mathbb{R}^{r\times p}$. Suppose that either W has full column rank or Π_{22} is nonsingular. Then there exists a $Z \in \mathcal{Z}_r(\Pi)$ such that $ZW = Y$ if and only if $\Pi \in \mathbf{\Pi}_{q,r}$ and $Y \in \mathcal{Z}_r(\Pi_W)$.*

Proof. To prove the 'if' statement, suppose that $\Pi \in \mathbf{\Pi}_{q,r}$ and $Y \in \mathcal{Z}_r(\Pi_W)$. By Theorem A.7 there exists a $Z \in \mathcal{Z}_r(\Pi)$ such that $ZW = Y$. To prove the 'only if' statement, suppose that there exists a $Z \in \mathcal{Z}_r(\Pi)$ satisfying $ZW = Y$. Therefore, $\mathcal{Z}_r(\Pi)$ is nonempty and (A.9) implies that $\Pi \,|\, \Pi_{22} \geqslant 0$. Consequently, $\Pi \in \mathbf{\Pi}_{q,r}$. Finally, $Y \in \mathcal{Z}_r(\Pi_W)$ follows directly from multiplying the defining quadratic matrix inequality from left by $W^\top$ and right by W. ☐

Corollary A.10. *Let $\Pi \in \mathbb{S}^{q+r}$ with $\Pi_{22} \leqslant 0$ and $\ker \Pi_{22} \subseteq \ker \Pi_{21}$. Consider $W \in \mathbb{R}^{q \times p}$ and $Y \in \mathbb{R}^{r \times p}$. Assume that W has full column rank. Then there exists a matrix $Z \in \mathcal{Z}_r^+(\Pi)$ satisfying $ZW = Y$ if and only if $\Pi \,|\, \Pi_{22} > 0$ and $Y \in \mathcal{Z}_r^+(\Pi_W)$.*

The proof of Corollary A.10 follows the same lines as that of Corollary A.9, but applies Theorem A.8 rather than Theorem A.7. It is therefore omitted.

A.3 Matrix S-lemma and Finsler's lemma

In this section we deal with the question under what conditions all solutions to one quadratic matrix inequality also satisfy another QMI. In other words, we aim at finding necessary and sufficient conditions for the inclusion $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r(M)$, where $M, N \in \mathbb{S}^{q+r}$. We will also consider this inclusion with $\mathcal{Z}_r^0(N)$ instead of $\mathcal{Z}_r(N)$, and for $\mathcal{Z}_r^+(M)$ replacing $\mathcal{Z}_r(M)$. This leads to non-strict and strict versions of Yakubovich's S-lemma and Finsler's lemma.

A.3.1 Recap of standard S-lemma and Finsler's lemma

For future reference, we will start with a brief recap of 'standard' (vector-valued) S-lemmas and Finsler's lemma. The idea behind all of these results is that certain implications involving quadratic inequalities and equalities can be characterized in terms of feasibility of linear matrix inequalities. The following statement is the S-lemma for non-strict inequalities, which was first proven by Yakubovich in the 1970s.

Proposition A.11 (S-lemma)**.** *Let $M, N \in \mathbb{S}^n$ and suppose that N has at least one positive eigenvalue. Then $x^\top M x \geqslant 0$ for all $x \in \mathbb{R}^n$ satisfying $x^\top N x \geqslant 0$ if and only if there exists a real number $\alpha \geqslant 0$ such that $M - \alpha N \geqslant 0$.*

Next, we recall a version of the S-lemma involving a strict inequality on $x^\top M x$.

Proposition A.12 (Strict S-lemma)**.** *Let $M, N \in \mathbb{S}^n$ and suppose that N has at least one positive eigenvalue. Then $x^\top M x > 0$ for all nonzero $x \in \mathbb{R}^n$ satisfying $x^\top N x \geqslant 0$ if and only if there exists a real number $\alpha \geqslant 0$ such that $M - \alpha N > 0$.*

Finally, we recall Finsler's lemma, which involves an *equality* $x^\top N x = 0$. We state the result for a strict inequality on $x^\top M x$. We note that also a non-strict version of the result exists, but this will not be used in this book.

Proposition A.13 (Finsler's lemma)**.** *Let $M, N \in \mathbb{S}^n$. Then $x^\top M x > 0$ for all nonzero $x \in \mathbb{R}^n$ satisfying $x^\top N x = 0$ if and only if there exists a real number $\alpha \in \mathbb{R}$ such that $M - \alpha N > 0$.*

A.3.2 Reduction of the matrix case to the vector case

Throughout this section, we will consider matrices $M, N \in \mathbb{S}^{q+r}$ partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \text{ and } N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}. \tag{A.24}$$

We will provide conditions under which the inclusion $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r(M)$ is equivalent to the *vector-valued* implication $x^\top N x \geqslant 0 \implies x^\top M x \geqslant 0$. This will provide an important building block in obtaining matrix versions of the S-lemma. To proceed, we will need the following lemmas.

Lemma A.14. *Let $S \in \mathbb{S}^n$ be positive semidefinite. Given a nonzero vector $x \in \mathbb{R}^n$, there exists a matrix $\bar{X} \in \mathbb{R}^{n\times(n-1)}$ such that $x^\top S \bar{X} = 0$ and $\begin{bmatrix} x & \bar{X} \end{bmatrix}$ is nonsingular.*

Proof. If $x^\top S = 0$ the statement is immediate. Thus, assume that $x^\top S \neq 0$. Let $\bar{X} \in \mathbb{R}^{n\times(n-1)}$ be a matrix whose columns form a basis for $\ker x^\top S$. If $\begin{bmatrix} x & \bar{X} \end{bmatrix}$ is singular, then $x \in \operatorname{im} \bar{X}$ and, hence, $x^\top S x = 0$. However, since S is symmetric and positive semidefinite, this implies that $x^\top S = 0$. This yields a contradiction, and we conclude that $\begin{bmatrix} x & \bar{X} \end{bmatrix}$ is nonsingular. This proves the lemma. □

Lemma A.15. *Let $N \in \mathbf{\Pi}_{q,r}$. Let $x \in \mathbb{R}^q$ and $y \in \mathbb{R}^r$ be vectors, with x nonzero, such that*

$$\begin{bmatrix} x \\ y \end{bmatrix}^\top N \begin{bmatrix} x \\ y \end{bmatrix} \geqslant 0.$$

Then there exists a matrix $Z \in \mathcal{Z}_r(N)$ such that $y = Zx$.

Proof. Since x is nonzero and $N \,|\, N_{22} \geqslant 0$, we conclude from Lemma A.14 that there exists a matrix $\bar{X} \in \mathbb{R}^{q\times(q-1)}$ such that $x^\top (N \,|\, N_{22})\bar{X} = 0$ and $\begin{bmatrix} x & \bar{X} \end{bmatrix}$ is nonsingular. Define the matrix $\bar{Y} := -N_{22}^\dagger N_{21} \bar{X}$. Note that

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} I & N_{12}N_{22}^\dagger \\ 0 & I \end{bmatrix} \begin{bmatrix} N \,|\, N_{22} & 0 \\ 0 & N_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ N_{22}^\dagger N_{21} & I \end{bmatrix}.$$

Therefore, we have

$$\begin{bmatrix} x \\ y \end{bmatrix}^\top N \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix}^\top N \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix} = \bar{X}^\top (N \,|\, N_{22}) \bar{X} \geqslant 0$$

since $N \,|\, N_{22} \geqslant 0$. The latter two results imply that

$$\begin{bmatrix} x & \bar{X} \\ y & \bar{Y} \end{bmatrix}^\top N \begin{bmatrix} x & \bar{X} \\ y & \bar{Y} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}^\top N \begin{bmatrix} x \\ y \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix}^\top N \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix} \end{bmatrix} \geqslant 0.$$

Recall that $\begin{bmatrix} x & \bar{X} \end{bmatrix}$ is nonsingular. Thus, the matrix $Z := \begin{bmatrix} y & \bar{Y} \end{bmatrix} \begin{bmatrix} x & \bar{X} \end{bmatrix}^{-1}$ is a member of $\mathcal{Z}_r(N)$. In addition, note that $y = Zx$. This proves the lemma. $\square$

The following theorem provides conditions under which the inclusions $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r(M)$ and $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r^+(M)$ are equivalent to their respective vector-valued implications.

Theorem A.16. *Let $M, N \in \mathbb{S}^{q+r}$ with $N \in \mathbf{\Pi}_{q,r}$.*

(a) *Assume that N has at least one positive eigenvalue. Then the following two statements are equivalent:*

  (i) $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r(M)$,

  (ii) $z^\top M z \geqslant 0$ *for all* $z \in \mathbb{R}^{q+r}$ *satisfying* $z^\top N z \geqslant 0$.

(b) *Assume that $N_{22} < 0$. Then the following two statements are equivalent:*

  (i) $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r^+(M)$,

  (ii) $z^\top M z > 0$ *for all nonzero* $z \in \mathbb{R}^{q+r}$ *satisfying* $z^\top N z \geqslant 0$.

Proof. We first prove that (a).(i) implies (a).(ii). Assume that (a).(i) holds but, on the contrary, (a).(ii) does not hold. This implies that there exist vectors $x \in \mathbb{R}^q$ and $y \in \mathbb{R}^r$, not both zero, such that

$$\begin{bmatrix} x \\ y \end{bmatrix}^\top N \begin{bmatrix} x \\ y \end{bmatrix} \geqslant 0 \text{ and } \begin{bmatrix} x \\ y \end{bmatrix}^\top M \begin{bmatrix} x \\ y \end{bmatrix} < 0. \tag{A.25}$$

We claim that there exists a pair (x, y) satisfying (A.25) with $x \neq 0$.

To see this, suppose that $x = 0$ and y satisfy (A.25). We will use these vectors to construct a new pair $(\tilde{x}, \tilde{y})$ satisfying (A.25) with $\tilde{x} \neq 0$. By the hypothesis that $N_{22} \leqslant 0$, we have that $N_{22} y = 0$. In addition, since $\ker N_{22} \subseteq \ker N_{12}$ we obtain

$$N \begin{bmatrix} 0 \\ y \end{bmatrix} = 0. \tag{A.26}$$

Let $\begin{bmatrix} \bar{x}^\top & \bar{y}^\top \end{bmatrix}^\top$ be an eigenvector of N corresponding to a positive eigenvalue λ. Note that $\bar{x} \neq 0$ because $N_{22} \leqslant 0$. By (A.26), we see that

$$\left(\begin{bmatrix} 0 \\ y \end{bmatrix} + \varepsilon \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \right)^\top N \left(\begin{bmatrix} 0 \\ y \end{bmatrix} + \varepsilon \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \right) = \varepsilon^2 \lambda \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}^\top \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \geqslant 0$$

for any $\varepsilon \in \mathbb{R}$. In addition,

$$\left(\begin{bmatrix} 0 \\ y \end{bmatrix} + \varepsilon \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \right)^\top M \left(\begin{bmatrix} 0 \\ y \end{bmatrix} + \varepsilon \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \right) < 0$$

if ε is sufficiently small. Therefore, for sufficiently small $\varepsilon \neq 0$, the pair $(\varepsilon\bar{x}, y + \varepsilon\bar{y})$ satisfies (A.25). As $\bar{x} \neq 0$, the pair $(\tilde{x}, \tilde{y}) := (\varepsilon\bar{x}, y + \varepsilon\bar{y})$ satisfies (A.25) with $\tilde{x} \neq 0$. Let (x, y) be such a pair. By Lemma A.15 there exists a matrix $Z \in \mathcal{Z}_r(N)$ satisfying $\tilde{y} = Z\tilde{x}$. By (A.25) we see that

$$\tilde{x}^\top \begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} \tilde{x} < 0$$

that is, $Z \notin \mathcal{Z}_r(M)$. This, however, contradicts the assumption that $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r(M)$. Therefore, we conclude that (a).(ii) holds.

Next, we prove that (b).(i) implies (b).(ii). Therefore, assume that (b).(i) holds but, on the contrary, (b).(ii) does not hold. This implies that there exist vectors $x \in \mathbb{R}^q$ and $y \in \mathbb{R}^r$, not both zero, such that

$$\begin{bmatrix} x \\ y \end{bmatrix}^\top N \begin{bmatrix} x \\ y \end{bmatrix} \geqslant 0 \text{ and } \begin{bmatrix} x \\ y \end{bmatrix}^\top M \begin{bmatrix} x \\ y \end{bmatrix} \leqslant 0. \tag{A.27}$$

This implies that $x \neq 0$. Indeed, if $x = 0$ then also $y = 0$ by the hypothesis that $N_{22} < 0$. Thus, by Lemma A.15 there exists a matrix $Z \in \mathcal{Z}_r(N)$ satisfying $y = Zx$. By (A.27) this implies that

$$x^\top \begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} x \leqslant 0$$

that is, $Z \notin \mathcal{Z}_r^+(M)$. This contradicts the hypothesis that $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r^+(M)$. This shows that (b).(ii) holds.

Next, we prove that (a).(ii) implies (a).(i). Suppose that $Z \in \mathcal{Z}_r(N)$. Then we have that

$$y^\top \begin{bmatrix} I \\ Z \end{bmatrix}^\top N \begin{bmatrix} I \\ Z \end{bmatrix} y \geqslant 0, \quad \text{and thus} \quad y^\top \begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} y \geqslant 0$$

for all $y \in \mathbb{R}^q$. In other words, $Z \in \mathcal{Z}_r(M)$. The proof that (b).(ii) implies (b).(i) is analogous and therefore omitted. This proves the theorem. □

A.3.3 Non-strict matrix S-lemma and Finsler's lemma

In the following theorem we apply the results of the previous section to establish a matrix version of the S-lemma.

Theorem A.17 (Matrix S-lemma). *Let $M, N \in \mathbb{S}^{q+r}$. If there exists a real $\alpha \geqslant 0$ such that $M - \alpha N \geqslant 0$, then $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r(M)$. Next, assume that $N \in \mathbf{\Pi}_{q,r}$ and N has at least one positive eigenvalue. Then $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r(M)$ if and only if there exists a real $\alpha \geqslant 0$ such that $M - \alpha N \geqslant 0$.*

Proof. The 'if' statements are obvious. We thus focus on proving the 'only if' part of the second statement. Assume that $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r(M)$. By Theorem A.16, $x^\top M x \geqslant 0$ for all $x \in \mathbb{R}^{q+r}$ satisfying $x^\top N x \geqslant 0$. Finally, by Lemma A.11, we conclude that there exists a scalar $\alpha \geqslant 0$ such that $M - \alpha N \geqslant 0$. □

Similar to the 'standard' S-lemma (Lemma A.11), we note that the matrix S-lemma requires N to have at least one positive eigenvalue, an assumption known as the *Slater condition*. It turns out, however, that under additional assumptions on M and N, we can state a theorem analogous to Theorem A.17 in the case where $N \in \mathbf{\Pi}_{q,r}$ has no positive eigenvalues, equivalently, $N \,|\, N_{22} = 0$. In this special case, $\mathcal{Z}_r(N) = \mathcal{Z}_r^0(N)$ which leads to a matrix version of Finsler's lemma.

Theorem A.18 (Matrix Finsler's lemma). *Let $M, N \in \mathbb{S}^{q+r}$. If there exists $\alpha \in \mathbb{R}$ such that $M - \alpha N \geqslant 0$ then $\mathcal{Z}_r^0(N) \subseteq \mathcal{Z}_r(M)$. Next, define $\Theta \in \mathbb{S}^q$ by*

$$\Theta := \begin{bmatrix} I \\ -N_{22}^\dagger N_{21} \end{bmatrix}^\top M \begin{bmatrix} I \\ -N_{22}^\dagger N_{21} \end{bmatrix}.$$

Assume that

(a) $M, N \in \mathbf{\Pi}_{q,r}$,

(b) $N \,|\, N_{22} = 0$, *and*

(c) $\ker \Theta \subseteq \ker M \,|\, M_{22}$.

Then $\mathcal{Z}_r^0(N) \subseteq \mathcal{Z}_r(M)$ if and only if there exists $\alpha \geqslant 0$ such that $M - \alpha N \geqslant 0$.

Proof. The 'if' statements are obvious. Now, assume that $\mathcal{Z}_r^0(N) \subseteq \mathcal{Z}_r(M)$. Let $Z \in \mathcal{Z}_r^0(N)$, $\xi \in \ker N_{22}$, and $\eta \in \mathbb{R}^q$ be a nonzero vector. By hypothesis, we have

$$Z + \gamma \xi \eta^\top \in \mathcal{Z}_r(M) \tag{A.28}$$

for all $\gamma \in \mathbb{R}$. Recall that $M \in \mathbf{\Pi}_{q,r}$ and therefore $M_{22} \leqslant 0$. This implies that $M_{22}\xi = 0$, for otherwise there exists a sufficiently large $\gamma \in \mathbb{R}$ that violates (A.28). We have thus proven that $\ker N_{22} \subseteq \ker M_{22}$. Next, define the matrix

$$T := \begin{bmatrix} I & 0 \\ -N_{22}^\dagger N_{21} & I \end{bmatrix}.$$

Note that

$$T^\top N T = \begin{bmatrix} 0 & 0 \\ 0 & N_{22} \end{bmatrix} \text{ and } T^\top M T = \begin{bmatrix} \Theta & M_{12} - N_{12} N_{22}^\dagger M_{22} \\ M_{21} - M_{22} N_{22}^\dagger N_{21} & M_{22} \end{bmatrix}.$$

This yields

$$T^\top(M-\alpha N)T = \begin{bmatrix} \Theta & M_{12}-N_{12}N_{22}^\dagger M_{22} \\ M_{21}-M_{22}N_{22}^\dagger N_{21} & M_{22}-\alpha N_{22} \end{bmatrix}. \tag{A.29}$$

Next, note that $\ker M_{22} \subseteq \ker M_{12}$ implies that

$$\Theta = M\,|\,M_{22} + (M_{22}^\dagger M_{21} - N_{22}^\dagger N_{21})^\top M_{22}(M_{22}^\dagger M_{21} - N_{22}^\dagger N_{21}) \tag{A.30}$$

and

$$M_{22}(M_{22}^\dagger M_{21} - N_{22}^\dagger N_{21}) = M_{21} - M_{22}N_{22}^\dagger N_{21}. \tag{A.31}$$

Since $-N_{22}^\dagger N_{21} \in \mathcal{Z}_r^0(N) \subseteq \mathcal{Z}_r(M)$, we have $\Theta \geqslant 0$. Therefore, since $M_{22} \leqslant 0$, (A.30) and (A.31) imply $\ker(M \mid M_{22}) = \ker\Theta \cap \ker(M_{21} - M_{22}N_{22}^\dagger N_{21})$. Therefore, by the hypothesis that $\ker\Theta \subseteq \ker(M\,|\,M_{22})$ we must have $\ker\Theta = \ker(M\,|\,M_{22})$, and it follows that $\ker\Theta = \ker(M_{21} - M_{22}N_{22}^\dagger N_{21})$. Consequently, by (A.29) and $\Theta \geqslant 0$, we see that $T^\top(M-\alpha N)T \geqslant 0$ if and only if

$$M_{22} - \alpha N_{22} - (M_{21} - M_{22}N_{22}^\dagger N_{21})\,\Theta^\dagger(M_{12} - N_{12}N_{22}^\dagger M_{22}) \geqslant 0. \tag{A.32}$$

Since $N_{22} \leqslant 0$ and $\ker N_{22} \subseteq \ker M_{22} \subseteq \ker M_{12}$, we conclude that there exists a sufficiently large $\alpha \geqslant 0$ such that (A.32) holds. This implies that there exists an $\alpha \geqslant 0$ such that $M - \alpha N \geqslant 0$. This proves the theorem. □

The assumption (c) on the matrix Θ is required in the sense that Theorem A.18 is, in general, not valid without it. We illustrate this as follows.

Example A.19. Suppose that

$$N = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Note that $M, N \in \mathbf{\Pi}_{1,1}$ and $N\,|\,N_{22} = 0$. In this case, $\Theta = 0$ and $M\,|\,M_{22} = 1$ so the assumption (c) of Theorem A.18 does not hold. In addition, we see that $\mathcal{Z}_1^0(N) = \{1\} \subseteq \mathcal{Z}_1(M)$. Nonetheless, there does not exist an $\alpha \geqslant 0$ such that $M - \alpha N \geqslant 0$. ■

A.3.4 Strict matrix S-lemma and Finsler's lemma

Subsequently, we consider *strict* versions of the above theorems. We focus on the case that the inequality involving M is strict while the one on N is nonstrict. The following theorem provides a strict matrix S-lemma in case N_{22} is negative definite.

Theorem A.20 (Strict matrix S-lemma). *Let $M, N \in \mathbb{S}^{q+r}$. If there exists a real $\alpha \geqslant 0$ such that $M - \alpha N > 0$, then $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r^+(M)$. Next, assume that $N \in \mathbf{\Pi}_{q,r}$ and $N_{22} < 0$. Then $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r^+(M)$ if and only if there exists a real $\alpha \geqslant 0$ such that $M - \alpha N > 0$.*

Proof. The 'if' parts are clear. Therefore, we focus on proving the 'only if' part of the second statement. Suppose that $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r^+(M)$. By Theorem A.16, we have that $x^\top Mx > 0$ for all nonzero $x \in \mathbb{R}^{q+r}$ satisfying $x^\top Nx \geqslant 0$. We now distinguish two cases. First suppose that N has at least one positive eigenvalue. Then, by Lemma A.12, there exists a real $\alpha \geqslant 0$ such that $M - \alpha N > 0$. Next, suppose that N does not have any positive eigenvalues, i.e., $N \leqslant 0$. We clearly have that $x^\top Mx > 0$ for all nonzero $x \in \mathbb{R}^{q+r}$ satisfying $x^\top Nx = 0$. Then, by Lemma A.13, there exists a real $\bar{\alpha} \in \mathbb{R}$ such that $M - \bar{\alpha} N > 0$. If $\bar{\alpha} \geqslant 0$ then we have $M - \alpha N > 0$ for $\alpha = \bar{\alpha}$. On the other hand, if $\bar{\alpha} < 0$ then $M > \bar{\alpha} N \geqslant 0$, so $M - \alpha N > 0$ for $\alpha = 0$. This proves the theorem. □

One can even prove a strict matrix S-lemma in the case that N_{22} is not necessarily negative definite, but under the extra assumptions that $M_{22} \leqslant 0$ and the Slater condition holds on N. It turns out, however, that in that case we need two real numbers $\alpha \geqslant 0$ and $\beta > 0$ to state a necessary and sufficient condition.

Theorem A.21 (Strict matrix S-lemma with α and β). *Let $M, N \in \mathbb{S}^{q+r}$. Then we have that $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r^+(M)$ if there exist scalars $\alpha \geqslant 0$ and $\beta > 0$ such that*

$$M - \alpha N \geqslant \begin{bmatrix} \beta I & 0 \\ 0 & 0 \end{bmatrix}. \tag{A.33}$$

Assume, in addition, that $N \in \mathbf{\Pi}_{q,r}$, $M_{22} \leqslant 0$ and N has at least one positive eigenvalue. Then $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r^+(M)$ if and only if there exist $\alpha \geqslant 0$ and $\beta > 0$ such that (A.33) holds.

Proof. Both 'if' statements are clear, so we focus on the 'only if' part. Assume that $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r^+(M)$. We will first prove that $\ker N_{22} \subseteq \ker M_{22}$ and $\ker N_{22} \subseteq \ker M_{12}$. Let $Z \in \mathcal{Z}_r(N)$ and $v \in \ker N_{22}$. In addition, select any nonzero vector $w \in \mathbb{R}^q$ and define $\hat{Z} := vw^\top$. Since $N \in \mathbf{\Pi}_{q,r}$, we have that $Z + \gamma \hat{Z} \in \mathcal{Z}_r(N)$ for all $\gamma \in \mathbb{R}$. Therefore $Z + \gamma \hat{Z} \in \mathcal{Z}_r^+(M)$. We write

$$0 < \begin{bmatrix} I \\ Z + \gamma \hat{Z} \end{bmatrix}^\top M \begin{bmatrix} I \\ Z + \gamma \hat{Z} \end{bmatrix} = \mathcal{L}(\gamma) + \gamma^2 (v^\top M_{22} v)\, ww^\top \tag{A.34}$$

where $\mathcal{L}(\gamma)$ is a matrix that depends affinely on γ. This implies that $M_{22} v = 0$. Indeed, if $M_{22} v \neq 0$ then $v^\top M_{22} v < 0$ and we can find a sufficiently large $\gamma \in \mathbb{R}$ that violates (A.34). We conclude that $\ker N_{22} \subseteq \ker M_{22}$. Next, let $v \in \ker N_{22}$

and define $\hat{Z} := -vv^\top M_{21}$. Since $v \in \ker M_{22}$, we can write

$$0 < \begin{bmatrix} I \\ Z + \gamma \hat{Z} \end{bmatrix}^\top M \begin{bmatrix} I \\ Z + \gamma \hat{Z} \end{bmatrix} = \begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} - 2\gamma\, M_{12} vv^\top M_{21}. \tag{A.35}$$

This implies that $M_{12}v = 0$, for otherwise we can select a sufficiently large $\gamma \in \mathbb{R}$ violating (A.35). Therefore, we conclude that $\ker N_{22} \subseteq \ker M_{12}$. Subsequently, we claim that there exists a scalar $\beta > 0$ such that

$$\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r^+ \left(M - \begin{bmatrix} \beta I & 0 \\ 0 & 0 \end{bmatrix} \right). \tag{A.36}$$

Suppose on the contrary that this claim is false. Then there exists a sequence $\{\beta_i\}$ such that $\beta_i \to 0$ $(i \to \infty)$ and for all i there exists $Z_i \in \mathcal{Z}_r(N)$ such that

$$Z_i \notin \mathcal{Z}_r^+ \left(M - \begin{bmatrix} \beta_i I & 0 \\ 0 & 0 \end{bmatrix} \right). \tag{A.37}$$

Define $\mathcal{V} := \{ Z \in \mathbb{R}^{r \times q} \mid N_{22} Z = 0 \}$. Write Z_i as $Z_i = Z_i^1 + Z_i^2$ where $Z_i^1 \in \mathcal{V}^\perp$ and $Z_i^2 \in \mathcal{V}$. Here $\mathcal{V}$ denotes the orthogonal complement of $\mathcal{V}$ with respect to the standard trace inner product on $\mathbb{R}^{r \times q}$. Since $\ker N_{22} \subseteq \ker N_{12}$ we see that $Z_i^1 \in \mathcal{Z}_r(N)$ for all i. Next, we claim that $\{Z_i^1\}$ is bounded. We will prove this by contradiction. Thus, assume that $\{Z_i^1\}$ is unbounded. Clearly, the sequence $\left\{ \frac{Z_i^1}{\|Z_i^1\|} \right\}$ is bounded. By Bolzano-Weierstrass, it thus has a convergent subsequence with limit, say Z_*. Note that

$$\frac{1}{\|Z_i^1\|^2} (N_{11} + N_{12} Z_i^1 + (N_{12} Z_i^1)^\top + (Z_i^1)^\top N_{22} Z_i^1) \geqslant 0.$$

By taking the limit along the subsequence as $i \to \infty$, we obtain $Z_*^\top N_{22} Z_* \geqslant 0$. Using the fact that $N_{22} \leqslant 0$, we conclude that $Z_* \in \mathcal{V}$. Since $Z_i^1 \in \mathcal{V}^\perp$ for all i, also $\frac{Z_i^1}{\|Z_i^1\|} \in \mathcal{V}^\perp$ and thus $Z_* \in \mathcal{V}^\perp$. Therefore, we conclude that both $Z_* \in \mathcal{V}$ and $Z_* \in \mathcal{V}^\perp$. That is, $Z_* = 0$. This is a contradiction as $\frac{Z_i^1}{\|Z_i^1\|}$ has norm 1 for all i. We conclude that the sequence $\{Z_i^1\}$ is bounded. It thus contains a convergent subsequence with limit, say Z_*. Note that $\mathcal{Z}_r(N)$ is closed and thus $Z_* \in \mathcal{Z}_r(N)$. Since $\ker N_{22} \subseteq \ker M_{22}$ and $\ker N_{22} \subseteq \ker M_{12}$, (A.37) implies that

$$Z_i^1 \notin \mathcal{Z}_r^+ \left(M - \begin{bmatrix} \beta_i I & 0 \\ 0 & 0 \end{bmatrix} \right)$$

for all i. We take the limit as $i \to \infty$ along a subsequence with limit Z_*, which yields $Z_* \notin \mathcal{Z}_r^+(M)$. However, since $Z_* \in \mathcal{Z}_r(N)$, this contradicts our hypothesis that $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r^+(M)$. Therefore, we conclude that there exists a $\beta > 0$ such that (A.36) holds. In particular, this implies that there exists $\beta > 0$ such that

$$\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r \left(M - \begin{bmatrix} \beta I & 0 \\ 0 & 0 \end{bmatrix} \right).$$

Finally, by Theorem A.17, there exists a scalar $\alpha \geqslant 0$ such that (A.33) holds. □

Next, we state a matrix Finsler's lemma in the case of a strict inequality.

Theorem A.22 (Strict matrix Finsler's lemma). *Let $M, N \in \mathbb{S}^{q+r}$. Then $\mathcal{Z}_r^0(N) \subseteq \mathcal{Z}_r^+(M)$ if there exist scalars $\alpha \in \mathbb{R}$ and $\beta > 0$ such that (A.33) holds. Next, assume that $N \in \mathbf{\Pi}_{q,r}$, $N \,|\, N_{22} = 0$ and $M_{22} \leqslant 0$. Then $\mathcal{Z}_r^0(N) \subseteq \mathcal{Z}_r^+(M)$ if and only if there exist $\alpha \geqslant 0$ and $\beta > 0$ such that (A.33) holds.*

Proof. The 'if' statements are obvious. To prove the 'only if' statement, assume that $\mathcal{Z}_r^0(N) \subseteq \mathcal{Z}_r^+(M)$. Since $N \,|\, N_{22} = 0$ and $N \in \mathbf{\Pi}_{q,r}$, we have that $N \leqslant 0$. This implies that $\mathcal{Z}_r^0(N) = \mathcal{Z}_r(N)$. Therefore, we also have that $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r^+(M)$. We can thus use the same argument as in the proof of Theorem A.17 to show that $\ker N_{22} \subseteq \ker M_{22}$ and $\ker N_{22} \subseteq \ker M_{12}$. Next, define the matrices

$$T := \begin{bmatrix} I & 0 \\ -N_{22}^\dagger N_{21} & I \end{bmatrix} \text{ and } \Theta := \begin{bmatrix} I \\ -N_{22}^\dagger N_{21} \end{bmatrix}^\top M \begin{bmatrix} I \\ -N_{22}^\dagger N_{21} \end{bmatrix}$$

and observe that

$$T^\top N T = \begin{bmatrix} 0 & 0 \\ 0 & N_{22} \end{bmatrix} \text{ and } T^\top M T = \begin{bmatrix} \Theta & M_{12} - N_{12} N_{22}^\dagger M_{22} \\ M_{21} - M_{22} N_{22}^\dagger N_{21} & M_{22} \end{bmatrix}.$$

Since $-N_{22}^\dagger N_{21} \in \mathcal{Z}_r^0(N) \subseteq \mathcal{Z}_r^+(M)$ we have $\Theta > 0$. Then obviously, there exists a real $\beta > 0$ so that $\Theta - \beta I > 0$. We have that

$$T^\top \left(M - \alpha N - \begin{bmatrix} \beta I & 0 \\ 0 & 0 \end{bmatrix} \right) T = \begin{bmatrix} \Theta - \beta I & M_{12} - N_{12} N_{22}^\dagger M_{22} \\ M_{21} - M_{22} N_{22}^\dagger N_{21} & M_{22} - \alpha N_{22} \end{bmatrix}.$$

Therefore it holds that $T^\top \left(M - \alpha N - \begin{bmatrix} \beta I & 0 \\ 0 & 0 \end{bmatrix} \right) T \geqslant 0$ if and only if

$$M_{22} - \alpha N_{22} - (M_{21} - M_{22} N_{22}^\dagger N_{21}) (\Theta - \beta I)^{-1} (M_{12} - N_{12} N_{22}^\dagger M_{22}) \geqslant 0. \quad \text{(A.38)}$$

Because $N_{22} \leqslant 0$, $\ker N_{22} \subseteq \ker M_{22}$ and $\ker N_{22} \subseteq \ker M_{12}$, there exists a sufficiently large $\alpha \geqslant 0$ such that (A.38) holds. This proves the statement. □

Finally, we note that it is possible to combine the strict versions of the matrix S-lemma and Finsler's lemma, Theorems A.21 and A.22, into one result. This results in the following corollary. Note the absence of the Slater condition on N.

Corollary A.23. *Let $M, N \in \mathbb{S}^{q+r}$. Then $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r^+(M)$ if there exist scalars $\alpha \geqslant 0$ and $\beta > 0$ such that (A.33) holds. Next, assume that $N \in \mathbf{\Pi}_{q,r}$ and $M_{22} \leqslant 0$. Then $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r^+(M)$ if and only if there exist $\alpha \geqslant 0$ and $\beta > 0$ such that (A.33) holds.*

Proof. Once again, the 'if' parts are clear. To prove the 'only if' statement, we distinguish the cases that N has at least one positive eigenvalue, and $N \leqslant 0$ (equivalently, $N \,|\, N_{22} = 0$). In the first case, Theorem A.21 is directly applicable, resulting in the existence of $\alpha \geqslant 0$ and $\beta > 0$ such that (A.33) holds. In the second case, $\mathcal{Z}_r(N) = \mathcal{Z}_r^0(N)$ and Theorem A.22 yields $\alpha \geqslant 0$ and $\beta > 0$ satisfying (A.33). □

A.4 Notes and references

The majority of the results in this chapter, such as the parameterizations (Theorem A.6), and the matrix versions of the S-lemma and Finsler's lemma (Theorems A.17, A.18, A.20, A.21, and A.22), are based on the paper [168]. Additional matrix versions of the S-lemma can be found in [169].

The parameterization of Theorem A.6 can be simplified under the additional assumption that $\Pi_{22} < 0$. Indeed, in this case the last term of (A.17) (depending on the matrix T) is zero. If $\Pi_{22} < 0$, Theorem A.6.(b) can also be proven using [155, Corollary 2.3.6] by defining the matrices $A = \Pi_{12}\Pi_{22}^{-1}$, $B = I$, $Q = \Pi \,|\, \Pi_{22}$, $R = -\Pi_{22}$ and $X = Z^\top$ in that result.

Corollary A.10 is intimately related to the so-called elimination lemma [70, 142]. In fact, in the case that Π is nonsingular and has r negative and q positive eigenvalues, Corollary A.10 can also be obtained from [142, Lem. A.2] by taking $P = -\Pi$, $A = I$, $B = W^\perp$ and $C = YW^\dagger$ where $W^\perp \in \mathbb{R}^{(q-p)\times q}$ is any full row rank matrix such that $W^\perp W = 0$.

The standard (non-strict) S-lemma in Proposition A.11 was first proven by Yakubovich in [195]. We also refer to the survey paper [130]. For the strict version in Proposition A.12, we refer to [195] and [26, p. 24]. Moreover, Finsler's lemma (Proposition A.13) is named after the German mathematician Paul Finsler, and was first proven in 1936 [53]. Also a non-strict version of this result exists, see e.g., [200], although for this version a so-called Slater condition is required.

Lemma A.15 was instrumental in proving the matrix versions of the S-lemma. It can be regarded as an extension of [143, Lemma A.2] to the set of matrices $\mathbf{\Pi}_{q,r}$. Indeed, instead of requiring that N satisfies $N_{22} < 0$ and $N_{11} - N_{12}N_{22}^{-1}N_{21} > 0$, we have merely assumed non-strict inequalities.

The proof of Lemma A.1 for the case $r = q = p$ is given in [22, Fact 5.10.19] and for the case $r = p$ in [132, Lem. 3]. In this chapter, we have provided a constructive proof for the case that r and p are not necessarily equal.

Some of the other matrix theoretical results in this chapter are also taken from the work of Bernstein [22]. Indeed, Lemma A.2 is a direct consequence of [22, Prop. 6.1.7]. Moreover, the fact that (A.4) holds if and only if $Z \in \mathcal{Z}_r(\Pi)$ follows from [22, Fact 8.15.28]. For a proof of Haynsworth inertia theorem (Lemma A.4) we refer to [22, Fact 6.5.5].

One of the steps of the proof of Lemma A.3 was to show that the $2q \times 2q$ partitioned matrix

$$\begin{bmatrix} 0 & I_q \\ I_q & \hat{\Pi}_{11} \end{bmatrix}$$

where $\hat{\Pi}_{11}$ is a symmetric matrix, has q positive and q negative eigenvalues. The proof of this fact was taken from Maddocks [101, Lem. 5.1].

Bibliography

[1] W. Aangenent, D. Kostic, B. de Jager, R. van de Molengraft, and M. Steinbuch. Data-based optimal control. In *Proceedings of the American Control Conference*, pages 1460–1465, June 2005.

[2] M.S. Ackerman and S. Gugercin. Frequency-based reduced models from purely time-domain data via data informativity. *arxiv.org/abs/2311.05012*, 2024.

[3] A. Allibhoy and J. Cortes. Data-based receding horizon control of linear network systems. *IEEE Control Systems Letters*, 5(4):1207–1212, 2021.

[4] M. Alsalti, J. Berberich, V. G. Lopez, F. Allgöwer, and M. A. Müller. Data-based system analysis and control of flat nonlinear systems. In *Proceedings of the IEEE Conference on Decision and Control*, pages 1484–1489, 2021.

[5] M. Alsalti, V. G. Lopez, J. Berberich, F. Allgöwer, and M. A. Müller. Data-based control of feedback linearizable systems. *IEEE Transactions on Automatic Control*, 68(11):7014–7021, 2023.

[6] B. D. O. Anderson and A. C. Antoulas. Rational interpolation and state-variable realizations. *Linear Algebra and its Applications*, 137:479–509, 1990.

[7] A. C. Antoulas. *Approximation of Large-Scale Dynamical Systems.* Advances in Design and Control, SIAM, Philadelphia, PA, 2005.

[8] A. C. Antoulas. The Loewner framework and transfer functions of singular/rectangular systems. *Applied Mathematics Letters*, 54:36–47, 2016.

[9] A. C. Antoulas and B. D. O. Anderson. On the scalar rational interpolation problem. *IMA Journal of Mathematical Control and Information*, 3(2-3):61–88, 1986.

[10] A. C. Antoulas, C. A. Beattie, and S. Gugercin. *Interpolatory Methods for Model Reduction.* SIAM, Philadelphia, PA, 2020.

[11] A. C. Antoulas and J. C. Willems. Minimal rational interpolation and Prony's method. In A. Bensoussan and J. L. Lions, editors, *Analysis and Optimization of Systems. Lecture Notes in Control and Information Sciences*, volume 144, pages 297–306. Springer, Berlin, Heidelberg, 1990.

[12] P. J. Antsaklis and A. N. Michel. *A Linear Systems Primer*. Birkhaüser Boston, 2007.

[13] A. Astolfi. Model reduction by moment matching for linear and nonlinear systems. *IEEE Transactions on Automatic Control*, 55(10):2321–2336, 2010.

[14] K. J. Åström and B. Wittenmark. *Adaptive Control*. Addison-Wesley, 1989.

[15] G. Baggio, D. S. Bassett, and F. Pasqualetti. Data-driven control of complex networks. *Nature Communications*, 12(1):1–13, 2021.

[16] C. Beattie and S. Gugercin. Realization-independent $\mathcal{H}_2$-approximation. In *Proceedings of the IEEE Conference on Decision and Control*, pages 4953–4958, 2012.

[17] J. Berberich, A. Iannelli, A. Padoan, J. Coulson, F. Dörfler, and F. Allgöwer. A quantitative and constructive proof of Willems' fundamental lemma and its implications. In *Proceedings of the American Control Conference*, pages 4155–4160, 2023.

[18] J. Berberich, A. Koch, C. W. Scherer, and F. Allgöwer. Robust data-driven state-feedback design. In *American Control Conference*, pages 1532–1538, 2020.

[19] J. Berberich, J. Kohler, M. A. Müller, and F. Allgöwer. Data-driven model predictive control with stability and robustness guarantees. *IEEE Transactions on Automatic Control*, 66(4):1702–1717, 2021.

[20] J. Berberich, C. W. Scherer, and F. Allgöwer. Combining prior knowledge and data for robust controller design. *IEEE Transactions on Automatic Control*, 68(8):4618–4633, 2023.

[21] J. Berberich, S. Wildhagen, M. Hertneck, and F. Allgöwer. Data-driven analysis and control of continuous-time systems under aperiodic sampling. *IFAC-PapersOnLine*, 54(7):210–215, 2021.

[22] D. S. Bernstein. *Matrix Mathematics: Theory, Facts, and Formulas*. Princeton University Press, 2011.

[23] A. Bisoffi, C. De Persis, and P. Tesi. Data-driven control via Petersen's lemma. *Automatica*, 145:110537, 2022.

[24] A. Bisoffi, L. Li, C. De Persis, and N. Monshizadeh. Controller synthesis for input-state data with measurement errors. *IEEE Control Systems Letters*, 8:1571–1576, 2024.

[25] X. Bombois, G. Scorletti, M. Gevers, P. M. J. Van den Hof, and R. Hildebrand. Least costly identification experiment for control. *Automatica*, 42(10):1651–1662, 2006.

[26] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. Studies in Applied Mathematics. Society for Industrial and Applied Mathematics, 1994.

[27] S. Boyd and L. Vandenberghe. *Introduction to Applied Linear Algebra: Vectors, Matrices, and Least Squares*. Cambridge University Press, 2018.

[28] V. Breschi, C. De Persis, S. Formentin, and P. Tesi. Direct data-driven model-reference control with Lyapunov stability guarantees. In *Proceedings of the IEEE Conference on Decision and Control*, pages 1456–1461, 2021.

[29] A. M. Burohman, B. Besselink, J. M. A. Scherpen, and M. K. Camlibel. From data to reduced-order models via generalized balanced truncation. *IEEE Transactions on Automatic Control*, 68(10):6160–6175, 2023.

[30] A. M. Burohman, B. Besselink, J. M. A. Scherpen, and M. K. Camlibel. From data to reduced-order models via moment matching. *Systems & Control Letters*, 194:105965, 2024.

[31] C. I. Byrnes and W. Lin. Losslessness, feedback equivalence, and the global stabilization of discrete-time nonlinear systems. *IEEE Transactions on Automatic Control*, 39(1):83–98, 1994.

[32] M. K. Camlibel and P. Rapisarda. Beyond the fundamental lemma: from finite time series to linear system. *arxiv.org/pdf/2405.18962*, 2024.

[33] M. K. Camlibel, H. J. van Waarde, and P. Rapisarda. The shortest experiment for linear system identification. *Systems & Control Letters*, 197:106045, 2025.

[34] M. C. Campi, A. Lecchini, and S. M. Savaresi. Virtual reference feedback tuning: a direct method for the design of feedback controllers. *Automatica*, 38(8):1337–1346, 2002.

[35] D. Carnevale, S. Galeani, M. Sassano, and A. Serrani. External models for output regulation based on moment estimation from input-output data. *IFAC-PapersOnLine*, 50(1):7777–7782, 2017.

[36] J. Coulson, J. Lygeros, and F. Dörfler. Data-enabled predictive control: In the shallows of the DeePC. In *Proceedings of the European Control Conference*, pages 307–312, June 2019.

[37] J. Coulson, J. Lygeros, and F. Dörfler. Distributionally robust chance constrained data-enabled predictive control. *IEEE Transactions on Automatic Control*, 67(7):3289–3304, 2022.

[38] J. Coulson, H. J. van Waarde, and F. Dörfler. Robust fundamental lemma for data-driven control. *Extended abstract in International Symposium on Mathematical Theory of Networks and Systems, available online: arxiv.org/abs/2205.06636*, 2022.

[39] J. Coulson, H. J. van Waarde, J. Lygeros, and F. Dörfler. A quantitative notion of persistency of excitation and the robust fundamental lemma. *IEEE Control Systems Letters*, 7:1243–1248, 2023.

[40] T. Dai and M. Sznaier. A moments based approach to designing MIMO data driven controllers for switched systems. In *Proceedings of the IEEE Conference on Decision and Control*, pages 5652–5657, 2018.

[41] J. Dancis. A quantitative formulation of Sylvester's law of inertia. III. *Linear Algebra and its Applications*, 80:141–158, 1986.

[42] E. J. Davison and A. Goldenberg. The robust control of a general servomechanism problem: the servo compensator. *Automatica*, 11:461–471, 1975.

[43] G. de Carolis, S. Galeani, and M. Sassano. Data driven, robust output regulation in finite time for LTI systems. *International Journal of Nonlinear and Robust Control*, pages 5997–6015, 2018.

[44] C. De Persis and P. Tesi. Formulas for data-driven control: Stabilization, optimality, and robustness. *IEEE Transactions on Automatic Control*, 65(3):909–924, March 2020.

[45] C. De Persis and P. Tesi. Low-complexity learning of linear quadratic regulators from noisy data. *Automatica*, 128:109548, 2021.

[46] F. Dörfler, P. Tesi, and C. De Persis. On the certainty-equivalence approach to direct data-driven LQR design. *IEEE Transactions on Automatic Control*, 68(12):7989–7996, 2023.

[47] J. C. Doyle, K. Glover, P. Khargonekar, and B. A. Francis. State space solutions to standard H_2 and H_∞control problems. *IEEE Transactions on Automatic Control*, 34:831–847, 1989.

[48] G. E. Dullerud and F. Paganini. *A Course in Robust Control Theory: A Convex Approach*, volume 36. Springer Science+Business Media, New York, NY, 2000.

[49] J. Eising and J. Cortes. When sampling works in data-driven control: Informativity for stabilization in continuous time. *IEEE Transactions on Automatic Control*, pages 1–8, 2024.

[50] J. Eising and H. L. Trentelman. Informativity of noisy data for structural properties of linear systems. *Systems & Control Letters*, 158:105058, 2021.

[51] T. Faulwasser, R. Ou, G. Pan, P. Schmitz, and K. Worthmann. Behavioral theory for stochastic systems? A data-driven journey from Willems to Wiener and back again. *Annual Reviews in Control*, 55:92–117, 2023.

[52] M. Ferizbegovic, H. Hjalmarsson, P. Mattsson, and T. B. Schön. Willems' fundamental lemma based on second-order moments. In *Proceedings of the IEEE Conference on Decision and Control*, pages 396–401, 2021.

[53] P. Finsler. Über das vorkommen definiter und semidefiniter formen in scharen quadratischer formen. *Commentarii Mathematici Helvetici*, 9:188–192, 1936.

[54] B. A. Francis. The linear multivariable regulator problem. *SIAM Journal on Control and Optimization*, 15(3):486–505, 1977.

[55] B. A. Francis and W. M. Wonham. The internal model principle for linear multivariable regulators. *Applied Mathematics and Optimization*, 2(4):380, 1975.

[56] X. Fu and K. You. Direct data-driven stabilization of nonlinear affine systems via the Koopman operator. In *Proceedings of the IEEE Conference on Decision and Control*, pages 2668–2673, 2022.

[57] K. Furuta and M. Wongsaisuwan. Discrete-time LQG dynamic controller design using plant Markov parameters. *Automatica*, 31(9):1317–1324, 1995.

[58] M. Gevers. Identification for control: From the early achievements to the revival of experiment design. *European Journal of Control*, 11(4):335–352, 2005.

[59] M. Gevers. A personal view of the development of system identification: A 30-year journey through an exciting field. *IEEE Control Systems Magazine*, 26(6):93–105, 2006.

[60] M. Gevers, A. S. Bazanella, X. Bombois, and L. Miskovic. Identification and the information matrix: How to get just sufficiently rich? *IEEE Transactions on Automatic Control*, 54(12), Dec 2009.

[61] M. Gevers, A. S. Bazanella, D. F. Coutinho, and S. Dasgupta. Identifiability and excitation of polynomial systems. In *Proceedings of the IEEE Conference on Decision and Control*, pages 4278–4283, Dec 2013.

[62] M. Gevers and L. Ljung. Optimal experiment designs with respect to the intended model application. *Automatica*, 22(5):543–554, 1986.

[63] G. R. Gonçalves da Silva, A. S. Bazanella, C. Lorenzi, and L. Campestrini. Data-driven LQR control design. *IEEE Control Systems Letters*, 3(1):180–185, 2019.

[64] G. C. Goodwin and R. L. Payne. *Dynamic System Identification: Experiment Design and Data Analysis.* Academic Press, 1977.

[65] I. V. Gosea and A. C. Antoulas. Stability preserving post-processing methods applied in the Loewner framework. In *Proceedings of the IEEE Workshop on Signal and Power Integrity*, pages 1–4, 2016.

[66] I. V. Gosea, S. Gugercin, and C. Beattie. Data-driven balancing of linear dynamical systems. *SIAM Journal on Scientific Computing*, 44(1):A554–A582, 2022.

[67] S. Gugercin and A. C. Antoulas. A survey of model reduction by balanced truncation and some new results. *International Journal of Control*, 77(8):748–766, 2004.

[68] M. Guo, C. De Persis, and P. Tesi. Data-driven stabilization of nonlinear polynomial systems with noisy data. *IEEE Transactions on Automatic Control*, 67(8):4210–4217, 2022.

[69] M. L. J. Hautus. Strong detectability and observers. *Linear Algebra and its Applications*, 50:353–368, 1983.

[70] A. Helmersson. IQC synthesis based on inertia constraints. *IFAC Proceedings Volumes*, 32(2):3361–3366, 1999.

[71] D. Hill and P. Moylan. The stability of nonlinear dissipative systems. *IEEE Transactions on Automatic Control*, 21(5):708–711, 1976.

[72] H. Hjalmarsson, M. Gevers, S. Gunnarsson, and O. Lequin. Iterative feedback tuning: theory and applications. *IEEE Control Systems Magazine*, 18(4):26–41, Aug 1998.

[73] H. Hjalmarsson, S. Gunnarsson, and M. Gevers. A convergent iterative restricted complexity control design scheme. In *Proceedings of the IEEE Conference on Decision and Control*, pages 1735–1740, Dec 1994.

[74] H. Huang, M. K. Camlibel, R. Carloni, and H. J. van Waarde. Data-driven stabilization of polynomial systems using density functions. *arxiv.org/abs/2503.07092*, 2025.

[75] H. Huang, M. K. Camlibel, R. Carloni, and H. J. van Waarde. Feedback stabilization of polynomial systems: From model-based to data-driven methods. *arxiv.org/pdf/2505.14457*, 2025.

[76] A. Iannelli and R. S. Smith. A multiobjective LQR synthesis approach to dual control for uncertain plants. *IEEE Control Systems Letters*, 4(4):952–957, 2020.

[77] A. Iannelli, M. Yin, and R. S. Smith. Experiment design for impulse response identification with signal matrix models. *IFAC-PapersOnLine*, 54(7):625–630, 2021.

[78] T. C. Ionescu. Two-sided time-domain moment matching for linear systems. *IEEE Transactions on Automatic Control*, 61(9):2632–2637, 2015.

[79] A. Isidori and C. I. Byrnes. Output regulation of nonlinear systems. *IEEE Transactions on Automatic Control*, 35(2):131–140, 1990.

[80] T. Iwasaki and S. Hara. Generalized KYP lemma: unified frequency domain inequalities with design applications. *IEEE Transactions on Automatic Control*, 50(1):41–59, 2005.

[81] T. Iwata, S.-I. Azuma, R. Ariizumi, and T. Asai. Data informativity for distributed positive stabilization. *IEEE Control Systems Letters*, 8:826–831, 2024.

[82] H. Jansson and H. Hjalmarsson. Input design via LMIs admitting frequency-wise model specifications in confidence regions. *IEEE Transactions on Automatic Control*, 50(10):1534–1549, 2005.

[83] J. Jiao, H. J. van Waarde, H. L. Trentelman, M. K. Camlibel, and S. Hirche. Data-driven output synchronization of heterogeneous leader-follower multi-agent systems. In *Proceedings of the IEEE Conference on Decision and Control*, pages 466–471, 2021.

[84] H. J. Jongsma, H. L. Trentelman, and M. K. Camlibel. Model reduction of networked multiagent systems by cycle removal. *IEEE Transactions on Automatic Control*, 63(3):657–671, 2017.

[85] E. Kaiser, J. N. Kutz, and S. L. Brunton. Data-driven discovery of Koopman eigenfunctions for control. *Machine Learning: Science and Technology*, 2(3):035023, June 2021.

[86] R. E. Kalman. Design of a self-optimizing control system. *Transactions of the American Society of Mechanical Engineers*, 80(2):468–477, 1958.

[87] T. Kato. *A Short Introduction to Perturbation Theory for Linear Operators*. Springer Berlin Heidelberg, 1982.

[88] J. Kenanian, A. Balkan, R. M. Jungers, and P. Tabuada. Data driven stability analysis of black-box switched linear systems. *Automatica*, 109:108533, 2019.

[89] A. Koch, J. Berberich, and F. Allgöwer. Provably robust verification of dissipativity properties from data. *IEEE Transactions on Automatic Control*, 67(8):4248–4255, 2022.

[90] C. Kojima and K. Takaba. A generalized Lyapunov stability theorem for discrete-time systems based on quadratic difference forms. In *Proceedings of the IEEE Conference on Decision and Control*, pages 2911–2916, 2005.

[91] C. Kojima and K. Takaba. An LMI condition for asymptotic stability of discrete-time system based on quadratic difference forms. In *Proceedings of the IEEE Conference on Computer Aided Control System Design*, pages 1139–1143, 2006.

[92] R. L. Kosut, M. K. Lau, and S. P. Boyd. Set-membership identification of systems with parametric and nonparametric uncertainty. *IEEE Transactions on Automatic Control*, 37(7):929–941, 1992.

[93] Y. Lian, R. Wang, and C. N. Jones. Koopman based data-driven predictive control. *arxiv.org/abs/2102.05122*, 2021.

[94] D. Liu, P. Yan, and Q. Wei. Data-based analysis of discrete-time linear systems in noisy environment: Controllability and observability. *Information Sciences*, 288:314–329, 2014.

[95] L. Ljung. On the consistency of prediction error identification methods. In Raman K. Mehra and Dimitri G. Lainiotis, editors, *System Identification Advances and Case Studies*, volume 126 of *Mathematics in Science and Engineering*, pages 121–164. Elsevier, 1976.

[96] L. Ljung. *System Identification: Theory for the User*. Prentice Hall information and system sciences series. Prentice Hall PTR, 1999.

[97] V. G. Lopez and M. A. Müller. On a continuous-time version of Willems' lemma. In *Proceedings of the Conference on Decision and Control*, pages 2759–2764, 2022.

[98] V. G. Lopez, M. A. Müller, and P. Rapisarda. An input-output continuous-time version of Willems' lemma. *IEEE Control Systems Letters*, 8:916–921, 2024.

[99] A. Luppi, C. De Persis, and P. Tesi. On data-driven stabilization of systems with nonlinearities satisfying quadratic constraints. *Systems & Control Letters*, 163:105206, 2022.

[100] A. I. Lur'e and V. N. Postnikov. On the theory of stability of control systems. *Applied Mathematics and Mechanics*, 8(3):283–286, 1944.

[101] J. H. Maddocks. Restricted quadratic forms, inertia theorems, and the Schur complement. *Linear Algebra and its Applications*, 108:1–36, 1988.

[102] I. Markovsky and F. Dörfler. Identifiability in the behavioral setting. *IEEE Transactions on Automatic Control*, 68(3):1667–1677, 2023.

[103] I. Markovsky, E. Prieto-Araujo, and F. Dörfler. On the persistency of excitation. *Automatica*, 147:110657, 2023.

[104] I. Markovsky and P. Rapisarda. Data-driven simulation and control. *International Journal of Control*, 81(12):1946–1959, 2008.

[105] I. Markovsky, J. C. Willems, P. Rapisarda, and B. L. M. De Moor. Algorithms for deterministic balanced subspace identification. *Automatica*, 41(5):755–766, 2005.

[106] T. Martin and F. Allgöwer. Dissipativity verification with guarantees for polynomial systems from noisy input-state data. In *Proceedings of the American Control Conference*, pages 3963–3968, 2021.

[107] T. M. Maupong, J. C. Mayo-Maldonado, and P. Rapisarda. On Lyapunov functions and data-driven dissipativity. *IFAC-PapersOnLine*, 50(1):7783–7788, 2017.

[108] A. J. Mayo and A. C. Antoulas. A framework for the solution of the generalized realization problem. *Linear Algebra and its Applications*, 425(2-3):634–662, 2007.

[109] T. J. Meijer, S. A. N. Nouwens, V. S. Dolk, and W. P. M. H. Heemels. A frequency-domain version of Willems' fundamental lemma. *arxiv.org/abs/2311.15284*, 2023.

[110] M. Milanese and A. Vicino. Optimal estimation theory for dynamic systems with set membership uncertainty: An overview. *Automatica*, 27(6):997–1009, 1991.

[111] J. Miller, T. Dai, M. Sznaier, and B. Shafai. Data-driven control of positive linear systems using linear programming. In *Proceedings of the IEEE Conference on Decision and Control*, pages 1588–1594, 2023.

[112] J. Miller and M. Sznaier. Data-driven gain scheduling control of linear parameter-varying systems using quadratic matrix inequalities. *IEEE Control Systems Letters*, 7:835–840, 2023.

[113] V. K. Mishra, I. Markovsky, and B. Grossmann. Data-driven tests for controllability. *IEEE Control Systems Letters*, 5(2):517–522, 2021.

[114] B. P. Molinari. On strong controllability and observability in linear multivariable control. *IEEE Transactions on Automatic Control*, 26:761–764, 1976.

[115] M. Moonen, B. L. M. De Moor, L. Vandenberghe, and J. Vandewalle. On- and off-line identification of linear state-space models. *International Journal of Control*, 49(1):219–232, 1989.

[116] B. Moore. Principal component analysis in linear systems: Controllability, observability, and model reduction. *IEEE Transactions on Automatic Control*, 26(1):17–32, 1981.

[117] A. S. Morse. Structural invariants of linear multivariable systems. *SIAM Journal on Control and Optimization*, 11:446–465, 1973.

[118] C. Mullis and R. Roberts. Synthesis of minimum roundoff noise fixed point digital filters. *IEEE Transactions on Circuits and Systems*, 23(9):551–562, 1976.

[119] H. Niu, H. Gao, and Z. Wang. A data-driven controllability measure for linear discrete-time systems. In *Proceedings of the IEEE Data Driven Control and Learning Systems Conference*, pages 455–466, May 2017.

[120] B. Nortmann and T. Mylvaganam. Direct data-driven control of linear time-varying systems. *IEEE Transactions on Automatic Control*, 68(8):4888–4895, 2023.

[121] A. Padoan and J. Coulson. Distances between finite-horizon linear behaviors. *arxiv.org/abs/2503.22849*, 2025.

[122] A. Padoan, J. Coulson, H. J. van Waarde, J. Lygeros, and F. Dörfler. Behavioral uncertainty quantification for data-driven control. In *Proceedings of the IEEE Conference on Decision and Control*, pages 4726–4731, 2022.

[123] A. Padoan, G. Scarciotti, and A. Astolfi. A geometric characterization of the persistence of excitation condition for the solutions of autonomous systems. *IEEE Transactions on Automatic Control*, 62(11):5666–5677, 2017.

[124] G. Pan, R. Ou, and T. Faulwasser. On a stochastic fundamental lemma and its use for data-driven optimal control. *IEEE Transactions on Automatic Control*, 68(10):5922–5937, 2023.

[125] U. S. Park and M. Ikeda. Stability analysis and control design of LTI discrete-time systems by the direct use of time series data. *Automatica*, 45(5):1265–1271, 2009.

[126] B. Peherstorfer and K. Willcox. Dynamic data-driven model reduction: adapting reduced models from incomplete data. *Advanced Modeling and Simulation in Engineering Sciences*, 3(1):1–22, 2016.

[127] G. Pipeleers and L. Vandenberghe. Generalized KYP lemma with real data. *IEEE Transactions on Automatic Control*, 56(12):2942–2946, 2011.

[128] M. Pirani and S. Sundaram. On the smallest eigenvalue of grounded Laplacian matrices. *IEEE Transactions on Automatic Control*, 61(2):509–514, 2016.

[129] J. W. Polderman and J. C. Willems. *Introduction to Mathematical Systems Theory: a behavioral approach.* Springer Verlag, 1997.

[130] I. Pólik and T. Terlaky. A survey of the S-lemma. *SIAM Review*, 49(3):371–418, 2007.

[131] V. M. Popov. *Hyperstability of Control Systems.* Springer-Verlag, New York, 1973.

[132] A. Rantzer. On the Kalman-Yakubovich-Popov lemma. *Systems & Control Letters*, 28(1):7–10, 1996.

[133] P. Rapisarda, M. K. Camlibel, and H. J. van Waarde. A fundamental lemma for continuous-time systems, with applications to data-driven simulation. *Systems & Control Letters*, 179:105603, 2023.

[134] P. Rapisarda, M. K. Camlibel, and H. J. van Waarde. A persistency of excitation condition for continuous-time systems. *IEEE Control Systems Letters*, 7:589–594, 2023.

[135] P. Rapisarda and H. L. Trentelman. Identification and data-driven model reduction of state-space representations of lossless and dissipative systems from noise-free data. *Automatica*, 47(8):1721–1728, 2011.

[136] P. Rapisarda, H. J. van Waarde, and M.K. Camlibel. Orthogonal polynomial bases for data-driven analysis and control of continuous-time systems. *IEEE Transactions on Automatic Control*, 69(7):4307–4319, 2024.

[137] A. Romer, J. Berberich, J. Köhler, and F. Allgöwer. One-shot verification of dissipativity properties from input-output data. *IEEE Control Systems Letters*, 3(3):709–714, July 2019.

[138] A. Romer, J. M. Montenbruck, and F. Allgöwer. Determining dissipation inequalities from input-output samples. *IFAC-PapersOnLine*, 50(1):7789–7794, 2017.

[139] A. Romer, J. M. Montenbruck, and F. Allgöwer. Sampling strategies for data-driven inference of passivity properties. In *Proceedings of the IEEE Conference on Decision and Control*, pages 6389–6394, 2017.

[140] T. E. Rosa and B. Jayawardhana. On the one-shot data-driven verification of dissipativity of LTI systems with general quadratic supply rate function. In *Proceedings of the European Control Conference*, pages 1291–1296, 2021.

[141] G. Scarciotti and A. Astolfi. Data-driven model reduction by moment matching for linear and nonlinear systems. *Automatica*, 79:340–351, 2017.

[142] C. W. Scherer. LPV control and full block multipliers. *Automatica*, 37(3):361–375, 2001.

[143] C. W. Scherer. Relaxations for robust linear matrix inequality problems with verifications for exactness. *SIAM Journal on Matrix Analysis and Applications*, 27(2):365–395, 2005.

[144] C. W. Scherer and S. Weiland. *Lecture Notes DISC Course on Linear Matrix Inequalities in Control.* 1999.

[145] P. Schmitz, T. Faulwasser, P. Rapisarda, and K. Worthmann. A continuous-time fundamental lemma and its application in data-driven optimal control. *Systems & Control Letters*, 194:105950, 2024.

[146] P. Schmitz, T. Faulwasser, and K. Worthmann. Willems' fundamental lemma for linear descriptor systems and its use for data-driven output-feedback MPC. *IEEE Control Systems Letters*, 6:2443–2448, 2022.

[147] J. M. Schumacher. On the structure of strongly controllable systems. *International Journal of Control*, 38:525–545, 1983.

[148] R. Sepulchre. Data-driven control: Part I [about this issue]. *IEEE Control Systems Magazine*, 43(5):4–7, 2023.

[149] R. Sepulchre. Data-driven control: Part II [about this issue]. *IEEE Control Systems Magazine*, 43(6):5–7, 2023.

[150] A. Shakouri, H. J. van Waarde, and M. K. Camlibel. A new perspective on Willems' fundamental lemma: Universality of persistently exciting inputs. *IEEE Control Systems Letters*, 9:583–588, 2025.

[151] X. Shang, J. Cortes, and Y. Zheng. Willems' fundamental lemma for nonlinear systems with Koopman linear embedding. *arxiv.org/pdf/2409.16389*, 2024.

[152] G. Shi and R.E. Skelton. Markov data-based LQG control. *ASME Journal of Dynamical Systems, Measurement, and Control*, 122(3):551–559, 1998.

[153] L.M. Silverman. Properties and application of inverse systems. *IEEE Transactions on Automatic Control*, 13:436–437, 1968.

[154] R. E. Skelton and G. Shi. The data-based LQG control problem. In *Proceedings of the IEEE Conference on Decision and Control*, pages 1447–1452, Dec 1994.

[155] R. E. Skelton, T. Iwasaki, and K. M. Grigoriadis. *A Unified Algebraic Approach to Linear Control Design.* Taylor and Francis, 1998.

[156] T. R. V. Steentjes, M. Lazar, and P. M. J. van den Hof. H_∞ performance analysis and distributed controller synthesis for interconnected linear systems from noisy input-state data. In *Proceedings of the IEEE Conference on Decision and Control*, pages 3723–3728, 2021.

[157] T. R. V. Steentjes, M. Lazar, and P. M. J. Van den Hof. On data-driven control: Informativity of noisy input-output data with cross-covariance bounds. *IEEE Control Systems Letters*, 6:2192–2197, 2022.

[158] R. Strässer, J. Berberich, and F. Allgöwer. Data-driven control of nonlinear systems: Beyond polynomial dynamics. In *Proceedings of the IEEE Conference on Decision and Control*, pages 4344–4351, 2021.

[159] Y. Tian and Y. Takane. Schur complements and Banachiewicz-Schur forms. *Electronic Journal of Linear Algebra*, 13:405–418, 2005.

[160] H. L. Trentelman, A. A. Stoorvogel, and M. Hautus. *Control Theory for Linear Systems.* Springer Verlag, London, UK, 2001.

[161] H. L. Trentelman, H. J. van Waarde, and M. K. Camlibel. An informativity approach to the data-driven algebraic regulator problem. *IEEE Transactions on Automatic Control*, 67(11):6227–6233, 2022.

[162] J. Umenberger, M. Ferizbegovic, T. B. Schön, and H. Hjalmarsson. Robust exploration in linear quadratic reinforcement learning. In *Advances in Neural Information Processing Systems 32*, pages 15336–15346. Curran Associates, Inc., 2019.

[163] P. M. J. van den Hof. Closed-loop issues in system identification. *Annual Reviews in Control*, 22:173–186, 1998.

[164] P. M. J. van Den Hof and R. J. P. Schrama. Identification and control – closed-loop issues. *Automatica*, 31(12):1751–1770, 1995. Trends in System Identification.

[165] P. van Overschee and B. L. M. de Moor. *Subspace Identification for Linear Systems: Theory, Implementation, Applications.* Kluwer Academic Publishers, 1996.

[166] H. J. van Waarde. Beyond persistent excitation: Online experiment design for data-driven modeling and control. *IEEE Control Systems Letters*, 6:319–324, 2022.

[167] H. J. van Waarde and M. K. Camlibel. A matrix Finsler's lemma with applications to data-driven control. In *Proceedings of the IEEE Conference on Decision and Control*, pages 5777–5782, 2021.

[168] H. J. van Waarde, M. K. Camlibel, J. Eising, and H. L. Trentelman. Quadratic matrix inequalities with applications to data-based control. *SIAM Journal on Control and Optimization*, 61(4):2251–2281, 2023.

[169] H. J. van Waarde, M. K. Camlibel, and M. Mesbahi. From noisy data to feedback controllers: Nonconservative design via a matrix S-lemma. *IEEE Transactions on Automatic Control*, 67(1):162–175, 2022.

[170] H. J. van Waarde, M. K. Camlibel, P. Rapisarda, and H. L. Trentelman. Data-driven dissipativity analysis: Application of the matrix S-lemma. *IEEE Control Systems Magazine*, 42(3):140–149, 2022.

[171] H. J. van Waarde, M. K. Camlibel, and H. L. Trentelman. Data-driven analysis and design beyond common Lyapunov functions. In *Proceedings of the IEEE Conference on Decision and Control*, pages 2783–2788, 2022.

[172] H. J. van Waarde, C. De Persis, M. K. Camlibel, and P. Tesi. Willems' fundamental lemma for state-space systems and its extension to multiple datasets. *IEEE Control Systems Letters*, 4(3):602–607, 2020.

[173] H. J. van Waarde, J. Eising, M. K. Camlibel, and H. L. Trentelman. The informativity approach to data-driven analysis and control. *IEEE Control Systems Magazine*, 43(6):32–66, 2023.

[174] H. J. van Waarde, J. Eising, M. K. Camlibel, and H. L. Trentelman. A behavioral approach to data-driven control with noisy input–output data. *IEEE Transactions on Automatic Control*, 69(2):813–827, 2024.

[175] H. J. van Waarde, J. Eising, H. L. Trentelman, and M. K. Camlibel. Data informativity: A new perspective on data-driven analysis and control. *IEEE Transactions on Automatic Control*, 65(11):4753–4768, 2020.

[176] H. J. van Waarde and M. Mesbahi. Data-driven parameterizations of suboptimal LQR and H_2 controllers. In *Proceedings of the IFAC World Congress*, 2020.

[177] M. Verhaegen and P. Dewilde. Subspace model identification part 1. the output-error state-space model identification class of algorithms. *International Journal of Control*, 56(5):1187–1210, 1992.

[178] M. Verhaegen and V. Verdult. *Filtering and System Identification: A Least Squares Approach*. Cambridge university press, 2007.

[179] C. Verhoek, H. S. Abbas, R. Tóth, and S. Haesaert. Data-driven predictive control for linear parameter-varying systems. *IFAC-PapersOnLine*, 54(8):101–108, 2021. Proceedings of the IFAC Workshop on Linear Parameter Varying Systems.

[180] C. Verhoek, R. Toth, S. Haesaert, and A. Koch. Fundamental lemma for data-driven analysis of linear parameter-varying systems. In *Proceedings of the IEEE Conference on Decision and Control*, pages 5040–5046, 2021.

[181] J. Wang, S. Baldi, and H. J. van Waarde. Necessary and sufficient conditions for data-driven model reference control. *IEEE Transactions on Automatic Control*, pages 1–8, 2024.

[182] Z. Wang and D. Liu. Data-based controllability and observability analysis of linear discrete-time systems. *IEEE Transactions on Neural Networks*, 22(12):2388–2392, Dec 2011.

[183] J. C. Willems. Least squares stationary optimal control and the algebraic Riccati equation. *IEEE Transactions on Automatic Control*, 16:621–634, 1971.

[184] J. C. Willems. Dissipative dynamical systems part I: general theory. *Archive for Rational Mechanics and Analysis*, 45:321–351, 1972.

[185] J. C. Willems. Dissipative dynamical systems Part II: linear systems with quadratic supply rates. *Archive for Rational Mechanics and Analysis*, 45:352–393, 1972.

[186] J. C. Willems. From time series to linear system–Part I. finite dimensional linear time invariant systems. *Automatica*, 22(5):561–580, 1986.

[187] J. C. Willems. From time series to linear system–Part II. exact modelling. *Automatica*, 22(6):675–694, 1986.

[188] J. C. Willems. From time series to linear system–Part III: Approximate modelling. *Automatica*, 23(1):87–115, 1987.

[189] J. C. Willems. Paradigms and puzzles in the theory of dynamical systems. *IEEE Transactions on Automatic Control*, 36(3):259–294, 1991.

[190] J. C. Willems, P. Rapisarda, I. Markovsky, and B. L. M. De Moor. A note on persistency of excitation. *Systems & Control Letters*, 54(4):325–329, 2005.

[191] J. C. Willems and H. L. Trentelman. On quadratic differential forms. *SIAM Journal on Control and Optimization*, 36(5):1703–1749, 1998.

[192] J. C. Willems and H. L. Trentelman. Synthesis of dissipative systems using quadratic differential forms: Part I. *IEEE Transactions on Automatic Control*, 47(1):53–69, 2002.

[193] J. C. Willems and H. L. Trentelman. Synthesis of dissipative systems using quadratic differential forms: Part II. *IEEE Transactions on Automatic Control*, 47(1):70–86, 2002.

[194] W.M. Wonham. *Linear Multivariable Control: a Geometric Approach*. Springer Verlag, New York, 1985.

[195] V. A. Yakubovich. S-procedure in nonlinear control theory. *Vestnik Leningrad University Mathematics*, 4:73–93, 1977.

[196] H. Yang and S. Li. A data-driven predictive controller design based on reduced Hankel matrix. In *Proceedings of the Asian Control Conference*, pages 1–7, 2015.

[197] M. Yin, A. Iannelli, and R. S. Smith. Maximum likelihood estimation in data-driven modeling and control. *IEEE Transactions on Automatic Control*, 68(1):317–328, 2023.

[198] Y. Yu, S. Talebi, H. J. van Waarde, U. Topcu, M. Mesbahi, and B. Acikmese. On controllability and persistency of excitation in data-driven control: Extensions of Willems' fundamental lemma. In *Proceedings of the IEEE Conference on Decision and Control*, pages 6485–6490, 2021.

[199] B. Zhou, Z. Wang, Y. Zhai, and H. Yuan. Data-driven analysis methods for controllability and observability of a class of discrete LTI systems with delays. In *Proceedings of the IEEE Data Driven Control and Learning Systems Conference*, pages 380–384, May 2018.

[200] Y. Zi-zong and G. Jin-hai. Some equivalent results with Yakubovich's S-Lemma. *SIAM Journal on Control and Optimization*, 48(7):4474–4480, 2010.

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