Data-based analysis and design beyond common Lyapunov functions

Henk van Waarde

Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence
and
Jan C. Willems Center for Systems and Control

University of Groningen

Joint research with Kanat Camlibel and Harry Trentelman

Topic of this talk

This talk is about assessing system properties and finding controllers from data

Some recent contributions:

Formulas for Data-Driven Control: Stabilization, Optimality, and Robustness
Claudio De Persis and Pietro Tesi

Robust data-driven state-feedback design
Julian Berberich[1], Anne Koch[1], Carsten W. Scherer[2], and Frank Allgöwer[1]

From Noisy Data to Feedback Controllers: Nonconservative Design via a Matrix S-Lemma
Henk J. van Waarde, M. Kanat Camlibel, *Member, IEEE*, and Mehran Mesbahi, *Fellow, IEEE*

Provably Robust Verification of Dissipativity Properties from Data
Anne Koch, Julian Berberich, and Frank Allgöwer

On Data-Driven Control: Informativity of Noisy Input-Output Data With Cross-Covariance Bounds
Tom R. V. Steentjes, *Graduate Student Member, IEEE*, Mircea Lazar, *Member, IEEE*, and Paul M. J. Van den Hof, *Fellow, IEEE*

- Lyapunov approaches with stability/dissipativity guarantees despite noisy data
- Solution often phrased in terms of data-based LMIs
- Common Lyapunov function for all systems consistent with data.

Can we design/analyze systems without the assumption of common Lyapunov functions?

Motivations:

- CLF is conservative in general
- Computation

Outline of the talk

1. Data-based stability, stabilizability and stabilization
2. Conditions with a common Lyapunov function
3. Beyond common Lyapunov functions
4. Conclusions

Data-based stability, stabilizability and stabilization

Data-based stability analysis

Consider the system

$$\boldsymbol{x}(t+1) = A_* \boldsymbol{x}(t) + \boldsymbol{w}(t),$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state and $\boldsymbol{w} \in \mathbb{R}^n$ denotes noise.

The real matrix A_* is unknown.

Data:

$$X = \begin{bmatrix} x(0) & x(1) & \cdots & x(T) \end{bmatrix}.$$

Shifted data:

$$X_- = \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix}$$
$$X_+ = \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix}.$$

Noise: matrix $W_- = \begin{bmatrix} w(0) & w(1) & \cdots & w(T-1) \end{bmatrix}$ is unknown but bounded:

$$\begin{bmatrix} I \\ W_-^\top \end{bmatrix}^\top \Phi \begin{bmatrix} I \\ W_-^\top \end{bmatrix} \geqslant 0.$$

for some known matrix $\Phi \in \mathbb{S}^{n+T}$ satisfying $\Phi_{22} < 0$ and $\Phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{21} \geqslant 0$.

Notion of informative data

stability analysis

The set of all systems explaining the data:

$$\Sigma_\mathsf{s} = \{A \in \mathbb{R}^{n\times n} \mid X_+ = AX_- + W_- \text{ for some } W_- \text{ satisfying bound}\}.$$

Note: $A \in \Sigma_\mathsf{s}$ if and only if

$$\begin{bmatrix} I \\ A^\top \end{bmatrix}^\top \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix}^\top \begin{bmatrix} I \\ A^\top \end{bmatrix} \geqslant 0.$$

Definition: The data X are called

1. informative for stability if every $A \in \Sigma_\mathsf{s}$ is Schur.
2. informative for quadratic stability if there exists a real matrix $P > 0$ such that $P - APA^\top > 0$ for all $A \in \Sigma_\mathsf{s}$.

Thus, informativity for quadratic stability implies that $x^\top P^{-1} x$ is a common Lyapunov function for all $A \in \Sigma_\mathsf{s}$.

Data-based stabilizability analysis and stabilization

Next, consider the system

$$\boldsymbol{x}(t+1) = A_* \boldsymbol{x}(t) + B_* \boldsymbol{u}(t) + \boldsymbol{w}(t),$$

where $\boldsymbol{u} \in \mathbb{R}^m$ is the input.

Data: state samples X and inputs $U_- = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix}$.

All systems explaining the data:

$$\Sigma_{\mathsf{i/s}} := \{(A,B) \mid X_+ = AX_- + BU_- + W_- \text{ for some } W_- \text{ satisfying the bound}\}.$$

Definition: The data (X, U_-) are called

1. informative for stabilizability if every $(A,B) \in \Sigma_{\mathsf{i/s}}$ is stabilizable.
2. informative for quadratic stabilizability if there exists a real matrix $P > 0$ such that $P - APA^\top + BB^\top > 0$ for all $(A,B) \in \Sigma_{\mathsf{i/s}}$.
3. informative for stabilization if there exists a $K \in \mathbb{R}^{m \times n}$ such that $A + BK$ is Schur for all $(A,B) \in \Sigma_{\mathsf{i/s}}$.
4. informative for quadratic stabilization if there exists a matrix $K \in \mathbb{R}^{m \times n}$ and a real matrix $P > 0$ such that $P - (A+BK)P(A+BK)^\top > 0$ for all $(A,B) \in \Sigma_{\mathsf{i/s}}$.

Conditions with a common Lyapunov function

Conditions for quadratic stability

LMI for robust stability

Proposition: The data X, generated by $\boldsymbol{x}(t+1) = A_* \boldsymbol{x}(t) + \boldsymbol{w}(t)$, are informative for quadratic stability if and only if there exists a real matrix $P > 0$ such that

$$\begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix}^\top > 0.$$

Interpretation: robust stability since for any $A \in \Sigma_{\mathsf{s}}$:

$$\underbrace{\begin{bmatrix} I \\ A^\top \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} \begin{bmatrix} I \\ A^\top \end{bmatrix}}_{\text{Lyapunov inequality}} - \underbrace{\begin{bmatrix} I \\ A^\top \end{bmatrix}^\top \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix}^\top \begin{bmatrix} I \\ A^\top \end{bmatrix}}_{\geqslant 0} > 0.$$

Only if direction via matrix S-lemma:

QUADRATIC MATRIX INEQUALITIES WITH APPLICATIONS TO DATA-BASED CONTROL

HENK J. VAN WAARDE, M. KANAT CAMLIBEL, JAAP EISING, AND HARRY L. TRENTELMAN

Abstract. This paper studies several problems related to quadratic matrix inequalities (QMI's), i.e., inequalities in the semidefinite ordering involving quadratic functions of matrix variables. In particular, we provide conditions under which the solution set of a QMI is nonempty, convex, bounded, or has nonempty interior. We also provide a parameterization of the solution set of a given QMI. In addition, we state projection results, that characterize a subset of "structured" solutions to a QMI. Thereafter, we derive matrix versions of the classical S-lemma and Finsler's lemma, that provide conditions under which all solutions to one QMI also satisfy

Conditions for quadratic stabilizability and stabilization

Simplifying assumption (can be removed): The input-state data, generated by $\boldsymbol{x}(t+1) = A_*\boldsymbol{x}(t) + B_*\boldsymbol{u}(t) + \boldsymbol{w}(t)$, satisfy $\operatorname{rank}\begin{bmatrix} X_-^\top & U_-^\top \end{bmatrix} = n+m$.

Proposition: The data (X, U_-) are informative for quadratic stabilizability if and only if there exists a real matrix $P > 0$ satisfying

$$\begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix}^\top > 0. \qquad (1)$$

Note: Exactly the same LMI as before! (but X depends on U_- now)

Proposition: Let $\Theta := \Phi_{12} + X_+\Phi_{22}$. The data (X, U_-) are informative for quadratic stabilization if and only if there exists a matrix P satisfying (1) and

$$P > \begin{bmatrix} I \\ X_+^\top \end{bmatrix}^\top \Phi \begin{bmatrix} I \\ X_+^\top \end{bmatrix} - \Theta \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top \left(\begin{bmatrix} X_- \\ U_- \end{bmatrix} \Phi_{22} \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top \right)^{-1} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \Theta^\top$$

Moreover, if P satisfies both inequalities then

$$K = \left(U_-(\Phi_{22} + \Theta^\top \Gamma^\dagger \Theta) X_-^\top\right) \left(X_-(\Phi_{22} + \Theta^\top \Gamma^\dagger \Theta) X_-^\top\right)^\dagger$$

is stabilizing for all systems $(A, B) \in \Sigma_{\mathsf{i/s}}$, where $\Gamma := P - \begin{bmatrix} I & X_+ \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \end{bmatrix}^\top$.

Beyond common Lyapunov functions

Beyond common Lyapunov functions

stability analysis without LMIs

Theorem: Define, for $\lambda \in \mathbb{C}$,

$$\Psi(\lambda) := \begin{bmatrix} I \\ (X_+ - \lambda X_-)^\top \end{bmatrix}^* \Phi \begin{bmatrix} I \\ (X_+ - \lambda X_-)^\top \end{bmatrix}.$$

Assume that $\Psi(1)$ is invertible and the matrix

$$\begin{bmatrix} 0 & \Psi(1)^{-1} \\ \Psi(-1) & 2(\Theta X_-^\top - X_- \Theta^\top)\Psi(1)^{-1} \end{bmatrix}$$

has no eigenvalues on the imaginary axis. Then the following are equivalent:

1. The data X are informative for quadratic stability.
2. The data X are informative for stability.
3. $\Psi(1) < 0$, X_- has full row rank, and the matrix $(X_- \Phi_{22} X_-^\top)^{-1} X_- \Theta^\top$ is Schur.

Note: Third condition is not phrased in terms of LMIs.

Proof relies on the KYP lemma.

Also possible to extend result to stabilizability analysis (but stronger conditions on data).

Beyond common Lyapunov functions

simple example

Consider the discrete-time consensus protocol with one stubborn agent:

$$\boldsymbol{x}(t+1) = \begin{bmatrix} I - aL_g & 0_{(n-1)\times 1} \\ 0_{1\times(n-1)} & 0_{1\times 1} \end{bmatrix} \boldsymbol{x}(t) + \boldsymbol{w}(t),$$

where L_g is the grounded Laplacian of an undirected cycle graph with $n = 500$ nodes.

The noise affects only node 1, and is bounded as $\|w(t)\| \leqslant \epsilon$.
Experiments for different ϵ, with $T = 3000$ samples each (random initial state).

ϵ	im. eigenvalues	$\Psi(1) < 0$	$\operatorname{rank} X_- = n$	Schur
0.10	100%	100%	100%	100%
0.15	100%	95%	100%	100%
0.20	100%	75%	100%	100%
0.25	100%	55%	100%	100%
0.30	100%	33%	100%	100%

Table: Percentage of trials in which the different conditions hold, for various levels of ϵ.

Example would already be challenging for LMI solvers (125000+ variables...)

Conclusions

Conclusions

summary and future work

1. Informativity for (quadratic) stability, stabilizability and stabilization
2. Striking similarity between conditions for quadratic properties
3. Under an eigenvalue condition, informativity for stability and quadratic stability are equivalent
4. New condition for informativity for stability, that does not rely on LMIs
5. Future goals: extend to stabilizability and stabilization.

Thank you!