On Quadratic Matrix Inequalities for data-driven control

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Quadratic matrix inequalities: basic definitions

The object of study is the QMI:

means: symmetric positive semidefinite

$$\begin{bmatrix} I_q \\ Z \end{bmatrix}^\top \underbrace{\Pi}_{\in \mathbb{S}^{q+r}} \begin{bmatrix} I_q \\ Z \end{bmatrix} \geq 0,$$

in the variable $Z \in \mathbb{R}^{r \times q}$, and the solution set

$$\mathcal{Z}_r(\Pi) := \left\{ Z \in \mathbb{R}^{r \times q} \;\middle|\; \begin{bmatrix} I_q \\ Z \end{bmatrix}^\top \Pi \begin{bmatrix} I_q \\ Z \end{bmatrix} \geq 0 \right\}.$$

Why should we care about these ??

A simple example:

Consider the system:

$$x(t+1) = A_s x(t) + B_s u(t) + w(t),$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input and $w(t) \in \mathbb{R}^n$ is noise. The matrices A_s and B_s are unknown.

Goal: Find a controller $u = Kx$ such that $A_s + B_s K$ is Schur, using the data:

$$X := \begin{pmatrix} x(0) & x(1) & \cdots & x(T) \end{pmatrix} \text{ and } U_- := \begin{pmatrix} u(0) & u(1) & \cdots & u(T-1) \end{pmatrix}.$$

Define $X_- = \begin{pmatrix} x(0) & x(1) & \cdots & x(T-1) \end{pmatrix}$ and $X_+ = \begin{pmatrix} x(1) & x(2) & \cdots & x(T) \end{pmatrix}$.

The noise matrix $W_- := \begin{pmatrix} w(0) & w(1) & \cdots & w(T-1) \end{pmatrix}$ is unknown.

How to model the noise?

Several recent papers use QMI descriptions of the noise:

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Formulas for Data-Driven Control: Stabilization, Optimality, and Robustness

Claudio De Persis and Pietro Tesi

Abstract—**In a paper by Willems *et al.*, it was shown that persistently exciting data can be used to represent the input–output behavior of a linear system. Based on this fundamental result, we derive a parametrization of linear feedback systems that paves the way to solve important control problems using data-dependent linear matrix in-**

the achievements obtained in system identification. A major challenge is how to incorporate data-dependent stability and performance requirements in the control design procedure.

A. Literature Review

$\longleftarrow \quad W_- W_-^T \leq \gamma X_+ X_+^T$

Robust data-driven state-feedback design

Julian Berberich[1], Anne Koch[1], Carsten W. Scherer[2], and Frank Allgöwer[1]

Abstract—**We consider the problem of designing robust state-feedback controllers for discrete-time linear time-invariant systems, based directly on measured data. The proposed design procedures require no model knowledge, but only a single open-loop data trajectory, which may be affected by noise. First, a data-driven characterization of the uncertain class of closed-**

contributions which consider this result in the context of data-driven system analysis and control, including dissipativity verification from measured data [10] or an extension of [9] to certain classes of nonlinear systems [11]. Moreover, the recent work [12] derives a simple data-dependent closed-loop

$\longleftarrow \quad \begin{bmatrix} I \\ W_- \end{bmatrix}^T \Psi \begin{bmatrix} I \\ W_- \end{bmatrix} \geq 0$

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From Noisy Data to Feedback Controllers: Nonconservative Design via a Matrix S-Lemma

Henk J. van Waarde, M. Kanat Camlibel, *Member, IEEE*, and Mehran Mesbahi, *Fellow, IEEE*

Abstract—**In this article, we propose a new method to obtain feedback controllers of an unknown dynamical system directly from noisy input/state data. The key ingredient of our design is a new matrix S-lemma that will be proven in this article. We provide both strict and nonstrict versions of this S-lemma, which are of interest in their own right.**

In fact, one of the unsolved problems is to come up with *nonconservative* control design strategies using only a finite number of data samples.

We will tackle this problem by providing necessary and sufficient conditions on noisy data, under which controllers can

$\longleftarrow \quad \begin{bmatrix} I \\ W_-^T \end{bmatrix}^T \Phi \begin{bmatrix} I \\ W_-^T \end{bmatrix} \geq 0$

Interpretation: Energy bound on noise: $\Phi_{12} = 0$, $\Phi_{22} = -I$ imply

$$W_- W_-^T = \sum_{t=0}^{T-1} w(t) w(t)^T \leq \Phi_{11}.$$

- $*$ sample bounds: $\|w(t)\|_2^2 \leq \varepsilon \quad \forall t = 0, 1, \ldots, T-1 \implies$ with $\Phi_{11} = \varepsilon T I$
- $*$ sample covariance bounds: $\frac{1}{T} W_- \left(I - \frac{1}{T} \mathbb{1}\mathbb{1}^T\right) W_-^T \leq \Phi_{11}$
  (take $\Phi_{12} = 0$, $\Phi_{22} = -\frac{1}{T}\left(I - \frac{1}{T}\mathbb{1}\mathbb{1}^T\right)$)
- $*$ noise-free data: $W_- = 0$ if $\Phi_{11} = 0$, $\Phi_{12} = 0$ and $\Phi_{22} = -I$.

The assumption on the noise leads to a set of "explaining" systems:

$$\Sigma := \{ (A,B) \mid X_+ = AX_- + BU_- + W_- \text{ for some } W_-^T \in \mathcal{Z}_T(\Phi) \}$$

In particular, $(A_s, B_s) \in \Sigma$ (but there may be many other systems)

$$\Sigma = \left\{ (A,B) \;\middle|\; \begin{bmatrix} I \\ A^T \\ B^T \end{bmatrix}^T \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^T \begin{bmatrix} I \\ A^T \\ B^T \end{bmatrix} \geqslant 0 \right\}$$

note: a QMI in the unknown A and B!

Since we cannot distinguish between (A_s, B_s) and any other $(A,B) \in \Sigma$, we want to control ALL systems in Σ.

Definition: The data (U_-, X) are informative for quadratic stabilization if there exists a $K \in \mathbb{R}^{m \times n}$ and $P > 0$ such that

$$P - (A+BK)P(A+BK)^T > 0$$

for all $(A,B) \in \Sigma$.

Interpretation: K stabilizes all systems in Σ with common Lyapunov function $x^T P^{-1} x$.

Note: The Lyapunov inequality may be written as

$$\begin{bmatrix} I \\ A^T \\ B^T \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & -\begin{pmatrix} I \\ K \end{pmatrix} P \begin{pmatrix} I \\ K \end{pmatrix}^T \end{bmatrix} \begin{bmatrix} I \\ A^T \\ B^T \end{bmatrix} > 0$$

(Also a QMI in A, B for fixed P and K!)

So the question is: when do there exist K and P s.t.

$$\mathcal{Z}_{n+m}\left(\begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^T\right) \subseteq \mathcal{Z}^+_{n+m}\left(\begin{bmatrix} P & 0 \\ 0 & -\begin{pmatrix} I \\ K \end{pmatrix} P \begin{pmatrix} I \\ K \end{pmatrix}^T \end{bmatrix}\right) ?$$

this means: solution set of strict QMI.

Other examples where QMIs appear:

Input-output systems:

$$y(t) + P_1^s y(t-1) + \dots + P_L^s y(t-L) = Q_0^s u(t) + Q_1^s u(t-1) + \dots + Q_L^s u(t-L) + w(t),$$

where $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^m$ and $w(t) \in \mathbb{R}^p$, $t = L, L+1, \dots, T$.

Hankel matrix:

$$\left[\begin{array}{cccc} u(0) & u(1) & \cdots & u(T-L) \\ y(0) & y(1) & \cdots & y(T-L) \\ \vdots & \vdots & & \vdots \\ u(L) & u(L+1) & \cdots & u(T) \\ \hline y(L) & y(L+1) & \cdots & y(T) \end{array}\right] =: \left[\begin{array}{c} H_1 \\ \hline H_2 \end{array}\right]$$

Unknown coefficients: $R = \begin{bmatrix} Q_L^s & -P_L^s & \cdots & Q_1^s & -P_1^s & Q_0^s \end{bmatrix}$.

$$H_2 = R H_1 + W_- \implies \begin{bmatrix} I \\ W_-^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ H_2^T & -H_1^T \end{bmatrix} \begin{bmatrix} I \\ R^T \end{bmatrix}.$$

$W_- = (w(L) \cdots w(T))$

Set of explaining systems: $\Sigma = \left\{ R \;\middle|\; \begin{bmatrix} I \\ R^T \end{bmatrix}^T \begin{bmatrix} I & 0 \\ H_2^T & -H_1^T \end{bmatrix} \Phi \begin{bmatrix} I & 0 \\ H_2^T & -H_1^T \end{bmatrix} \begin{bmatrix} I \\ R^T \end{bmatrix} \geq 0 \right\}$

Lur'e systems:

$$\begin{aligned} x(t+1) &= A_s x(t) + B_s u(t) + E\varphi(y(t)) + w(t) \\ y(t) &= C_s x(t) + D_s u(t) + v(t) \end{aligned}$$

Data: X, U_- as before and $Y_- = \begin{pmatrix} y(0) & y(1) & \cdots & y(T-1) \end{pmatrix}$,
$F_- := \begin{pmatrix} \varphi(y(0)) & \varphi(y(1)) & \cdots & \varphi(y(T-1)) \end{pmatrix}$.

Noise: $W_- = \begin{pmatrix} w(0) & w(1) & \cdots & w(T-1) \\ v(0) & v(1) & \cdots & v(T-1) \end{pmatrix}$.

Set of explaining systems:

$$\Sigma = \left\{ (A,B,C,D) \;\middle|\; \begin{bmatrix} X_+ - EF_- \\ Y_- \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = W_- \text{ for some } W_-^T \in \mathcal{Z}_T(\Phi) \right\}.$$

again governed by a QMI!

Purpose of this talk:

Understand properties of QMIs in the context of data-driven (robust) control

QUADRATIC MATRIX INEQUALITIES WITH APPLICATIONS TO DATA-BASED CONTROL*

HENK J. VAN WAARDE†, M. KANAT CAMLIBEL†, JAAP EISING‡, AND HARRY L. TRENTELMAN†

Abstract. This paper studies several problems related to quadratic matrix inequalities (QMI's), i.e., inequalities in the Loewner order involving quadratic functions of matrix variables. In particular, we provide conditions under which the solution set of a QMI is nonempty, convex, bounded, or has nonempty interior. We also provide a parameterization of the solution set of a given QMI. In addition, we state results regarding the image of such sets under linear maps, which characterize a subset of "structured" solutions to a QMI. Thereafter, we derive matrix versions of the classical S-lemma and Finsler's lemma,

(submitted to SICON)

Quadratic matrix inequalities: basic properties

First order of business: when does there exist a solution $Z \in \mathbb{R}^{r \times q}$ to

$$\begin{bmatrix} I_q \\ Z \end{bmatrix}^T \underbrace{\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}}_{\in \mathbb{S}^{q+r}} \begin{bmatrix} I_q \\ Z \end{bmatrix} \geq 0 \quad ??$$

In general, not so easy: $Z \in \mathcal{Z}_r(\Pi) \iff \begin{bmatrix} \Pi_{11} & \Pi_{12} & -Z^T \\ \Pi_{21} & \Pi_{22} & I \\ -Z & I & 0 \end{bmatrix}$ has r negative eigenvalues. Not that useful...

We can say more if $\ker \Pi_{22} \subseteq \ker \Pi_{12}$ and $\Pi_{22} \leq 0$. Then:

Schur complement: $\Pi_{11} - \Pi_{12}\Pi_{22}^{\dagger}\Pi_{21}$

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} = \begin{bmatrix} I_q & \Pi_{12}\Pi_{22}^{\dagger} \\ 0 & I_r \end{bmatrix} \begin{bmatrix} \Pi | \Pi_{22} & 0 \\ 0 & \Pi_{22} \end{bmatrix} \begin{bmatrix} I_q & 0 \\ \Pi_{22}^{\dagger}\Pi_{21} & I_r \end{bmatrix}$$

Moore-Penrose pseudo-inverse.

Hence, for any $Z \in \mathbb{R}^{r \times q}$:

$$\begin{bmatrix} I \\ Z \end{bmatrix}^T \Pi \begin{bmatrix} I \\ Z \end{bmatrix} = \Pi | \Pi_{22} + (\Pi_{22}^{\dagger}\Pi_{21} + Z)^T \Pi_{22} (\Pi_{22}^{\dagger}\Pi_{21} + Z) \leq \Pi | \Pi_{22}$$

So a necessary condition for nonemptiness of $\mathcal{Z}_r(\Pi)$: $\Pi | \Pi_{22} \geq 0$.

Also sufficient because $\begin{bmatrix} I \\ -\Pi_{22}^{\dagger}\Pi_{21} \end{bmatrix}^T \Pi \begin{bmatrix} I \\ -\Pi_{22}^{\dagger}\Pi_{21} \end{bmatrix} = \Pi | \Pi_{22}$.

this is a solution if $\Pi | \Pi_{22} \geq 0$.

This motivates the class of "Π-matrices":

$$\mathbf{\Pi}_{q,r} := \left\{ \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \in \mathbb{S}^{q+r} \;\middle|\; \Pi_{22} \preceq 0,\ \ker \Pi_{22} \subseteq \ker \Pi_{12},\ \Pi | \Pi_{22} \geq 0 \right\}$$

Theorem: Let $\Pi \in \mathbf{\Pi}_{q,r}$. Then $\mathcal{Z}_r(\Pi)$:

a) is nonempty and convex

b) is bounded $\iff \Pi_{22} \prec 0$

c) has nonempty interior $\iff \Pi_{22} = 0$ or $\Pi | \Pi_{22} > 0$.

Now onto more interesting stuff...

Inclusions of solution sets of QMIs and matrix S-lemmas

Let $M, N \in \mathbb{S}^{q+r}$.

Problem: Under which conditions do we have that $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r(M)$?
(i.e., all solutions to one QMI satisfy another QMI).

A classical result (Yakubovich's S-lemma):

Let $M, N \in \mathbb{S}^n$ and suppose that N has at least one positive eigenvalue.
Then $x \in \mathbb{R}^n$ and $x^T N x \geq 0 \Rightarrow x^T M x \geq 0$ if and only if there exists $\alpha \geq 0$ such that $M - \alpha N \geq 0$.

Note: Here we need an S-lemma for matrix variables (and Loewner order)

Theorem (matrix S-lemma): Let $M \in \mathbb{S}^{q+r}$ and $N \in \Pi_{q,r}$ and suppose that N has at least one positive eigenvalue.
Then $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r(M) \iff \exists\, \alpha \geq 0$ such that $M - \alpha N \geq 0$.

So checking QMI inclusions boils down to feasibility of linear matrix inequalities!

Also possible to derive matrix S-lemmas with one strict inequality:

Theorem: Let $M, N \in \mathbb{S}^{q+r}$ and assume that $N \in \mathbf{\Pi}_{q,r}$ and $M_{22} \leq 0$.
Then $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r^+(M) \iff$ there exist $\alpha \geq 0$ and $\beta > 0$ such that:
$M - \alpha N \geq \begin{bmatrix} \beta I & 0 \\ 0 & 0 \end{bmatrix}$.

Things simplify if $N_{22} < 0$:

Theorem: Let $M, N \in \mathbb{S}^{q+r}$ and assume that $N \in \mathbf{\Pi}_{q,r}$ and $N_{22} < 0$.
Then $\mathcal{Z}_r(N) \subseteq \mathcal{Z}_r^+(M) \iff \exists\, \alpha \geq 0 :$ $M - \alpha N > 0$.

Now: how to apply this to data-driven control (e.g. stabilization)?

Recall that the input-state data (U_-, X) are informative for stabilization if and only if there exist $P>0$ and K s.t.

$$\mathcal{Z}_{n+m}\left(\begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^T\right) \subseteq \mathcal{Z}^+_{n+m}\left(\begin{bmatrix} P & 0 \\ 0 & -\begin{pmatrix} I \\ K \end{pmatrix} P \begin{pmatrix} I \\ K \end{pmatrix}^T \end{bmatrix}\right).$$

assumed to be in set $\Pi_{n,T}$

$$\Updownarrow \quad \text{By matrix S-lemma}$$

There exist $P>0$, $K \in \mathbb{R}^{m \times n}$ and scalars $\alpha \geq 0$, $\beta > 0$ s.t.

$$\begin{bmatrix} P-\beta I & 0 \\ 0 & -\begin{pmatrix} I \\ K \end{pmatrix} P \begin{pmatrix} I \\ K \end{pmatrix}^T \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^T \geq 0.$$

Observation: We can scale P and β by $\frac{1}{\alpha}$ and apply a change of variables and Schur complement to obtain the equivalent LMI condition:

The data (U_-, X) are informative for quadratic stabilization

$\iff$ There exist $P>0$, $L \in \mathbb{R}^{m \times n}$ and $\beta > 0$ s.t.

$$\begin{bmatrix} P-\beta I & 0 & 0 & 0 \\ 0 & 0 & 0 & P \\ 0 & 0 & 0 & L \\ 0 & P & L^T & P \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \end{bmatrix}^T \geq 0.$$

Here K and L are related by $L = KP$.

Remarks: The matrix S-lemmas presented here operate under mild assumptions ($N \in \Pi_{q,r}$), in comparison to:

* A matrix S-lemma derived from full-block S-procedure, by C. Scherer.
* A closely related result known as Petersen's lemma.

The new matrix S-lemmas enable:

* general necessary & sufficient conditions for informativity (without making a priori assumptions on the data).
* general noise models $\phi \in \Pi_{n,T}$

e.g. cross-covariance bounds →

On data-driven control: informativity of noisy input-output data with cross-covariance bounds

Tom R.V. Steentjes, Mircea Lazar, Paul M.J. Van den Hof

Abstract—**In this paper we develop new data informativity based controller synthesis methods that extend existing** knowledge on the noise in the form of quadratic bounds, via a matrix variant of the S-procedure. Quadratic noise bounds play

* treatment of noise in a subspace of the state-space: $\omega(t) \in \operatorname{im} E \quad \forall\, t = 0, 1, \ldots, T-1.$

But there is more to tell...

Image of $\mathcal{Z}_r(\Pi)$ under linear maps

Question: Let $\Pi \in \boldsymbol{\Pi}_{q,r}$, and $W \in \mathbb{R}^{q \times p}$. Can we characterize the set of matrices $\mathcal{Z}_r(\Pi)W \subseteq \mathbb{R}^{r \times p}$?

YES, using another QMI!

Theorem: Assume that either $\Pi_{22} < 0$ or W has full column rank. Then $\mathcal{Z}_r(\Pi)W = \mathcal{Z}_r(\Pi_W)$, where

$$\Pi_W := \begin{pmatrix} W^T \Pi_{11} W & W^T \Pi_{12} \\ \Pi_{21} W & \Pi_{22} \end{pmatrix}.$$

So the image of a QMI solution set under a linear map is again a solution set of a (different) QMI.

Corollary: Let $\Pi \in \mathbb{S}^{q+r}$ with $\Pi_{22} \leq 0$ and $\ker \Pi_{22} \subseteq \ker \Pi_{21}$. Consider $W \in \mathbb{R}^{q \times p}$ and $Y \in \mathbb{R}^{r \times p}$. Suppose that W has full column rank or $\Pi_{22} < 0$. Then $\exists Z \in \mathcal{Z}_r(\Pi)$ such that $ZW = Y$ if and only if $\Pi \in \boldsymbol{\Pi}_{q,r}$ and $Y \in \mathcal{Z}_r(\Pi_W)$.

Applications in data-driven control:

* Noise within a subspace:

Let $\hat{\phi} \in \Pi_{d,T}$ with $\hat{\phi}_{22} < 0$. Let $E \in \mathbb{R}^{n \times d}$.

Then $W_- = E\hat{W}_-$ for some $\hat{W}_-^T \in \mathcal{Z}_T(\hat{\phi})$ if and only if $W_-^T \in \mathcal{Z}_T(\phi)$, where

$$\phi := \begin{pmatrix} E\hat{\phi}_{11}E^T & E\hat{\phi}_{12} \\ \hat{\phi}_{21}E^T & \hat{\phi}_{22} \end{pmatrix}.$$

so we can capture the constraint $\operatorname{im} W_- \subseteq \operatorname{im} E$ in the noise model ϕ.

* Input-output systems:

$$y(t) + P_1^s y(t-1) + \dots + P_L^s y(t-L) = Q_0^s u(t) + Q_1^s u(t-1) + \dots + Q_L^s u(t-L) + w(t)$$

A (structured) state-space representation:

$$\begin{bmatrix} u(t-L+1) \\ y(t-L+1) \\ \vdots \\ u(t) \\ y(t) \end{bmatrix} = \underbrace{\left[\begin{array}{ccccc} \multicolumn{5}{c}{\text{Known matrix}} \\ \hline Q_L^s & -P_L^s & \cdots & Q_1^s & -P_1^s \end{array}\right]}_{=:A_s} \underbrace{\begin{bmatrix} u(t-L) \\ y(t-L) \\ \vdots \\ u(t-1) \\ y(t-1) \end{bmatrix}}_{=:x(t)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ I \\ Q_0^s \end{bmatrix}}_{=:B_s} u(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix} w(t)$$

As also noted here, $\rightarrow$

(A_s, B_s) are structured!

Combining Prior Knowledge and Data for Robust Controller Design

Julian Berberich[1], Carsten W. Scherer[2], and Frank Allgöwer[1]

Abstract—**We present a framework for systematically combining data of an unknown linear time-invariant system with prior knowledge on the system matrices or on the uncertainty for robust controller design. Our approach leads to linear matrix inequality (LMI) based feasibility criteria which guarantee stability**

data is an important and largely open problem. In this paper, we present a framework for combining prior knowledge and noisy data of a linear time-invariant (LTI) system for controller design based on robust control theory.

Suppose that we want to design a dynamic output feedback:

$$u(t) = Kx(t)$$

$$= K \begin{pmatrix} u(t-L) \\ y(t-L) \\ \vdots \\ u(t-1) \\ y(t-1) \end{pmatrix}.$$

known unknown coefficients

Uncertain (A,B)'s are of the form $(A \quad B) = \begin{pmatrix} F \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ R \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} A^T \\ B^T \end{pmatrix} = (F^T \quad 0) + R^T (0 \quad I).$$

As before, the unknown coefficients satisfy

$$R^T \in \mathcal{Z}_{(p+m)L+m}\left(\underbrace{\begin{bmatrix} I & 0 \\ H_2^T & -H_1^T \end{bmatrix} \Phi \begin{bmatrix} I & 0 \\ H_2^T & -H_1^T \end{bmatrix}}_{N}\right).$$

We want that $P - (A \quad B)\begin{pmatrix} I \\ K \end{pmatrix} P \begin{pmatrix} I \\ K \end{pmatrix}^T \begin{pmatrix} A^T \\ B^T \end{pmatrix} > 0$, that is,

$$\begin{bmatrix} I \\ R^T(0 \; I) \end{bmatrix} \underbrace{\begin{bmatrix} P - \begin{pmatrix} F \\ 0 \end{pmatrix}\begin{pmatrix} I \\ K \end{pmatrix} P \begin{pmatrix} I \\ K \end{pmatrix}^T \begin{pmatrix} F \\ 0 \end{pmatrix}^T & -\begin{pmatrix} F \\ 0 \end{pmatrix}\begin{pmatrix} I \\ K \end{pmatrix} P \begin{pmatrix} I \\ K \end{pmatrix}^T \\ -\begin{pmatrix} I \\ K \end{pmatrix} P \begin{pmatrix} I \\ K \end{pmatrix}^T \begin{pmatrix} F \\ 0 \end{pmatrix}^T & -\begin{pmatrix} I \\ K \end{pmatrix} P \begin{pmatrix} I \\ K \end{pmatrix}^T \end{bmatrix}}_{M} \begin{bmatrix} I \\ R^T(0 \; I) \end{bmatrix} > 0$$

We thus want: $R^T \in \mathcal{Z}_{(p+m)L+m}(N) \Rightarrow R^T \begin{pmatrix} 0 & I \end{pmatrix} \in \mathcal{Z}^+_{(p+m)L+m}(M)$

Equivalently, $\mathcal{Z}_{(p+m)L+m}(N) \begin{pmatrix} 0 & I \end{pmatrix} \subseteq \mathcal{Z}^+_{(p+m)L+m}(M)$.

But this is again a QMI solution set!

Rest follows again from matrix S-lemma.

A behavioral approach is also available that avoids state-space representations:

A behavioral approach to data-driven control with noisy input-output data

H.J. van Waarde, J. Eising, M.K. Camlibel, *Senior Member, IEEE*, and H.L. Trentelman, *Life Fellow, IEEE*

***Abstract*—This paper deals with data-driven stability analysis and feedback stabilization of linear input-output systems in autoregressive (AR) form. We assume that noisy input-output data on a finite time-interval have been obtained from some unknown AR system. Data-based tests**

applicable. A potential downside of this approach is that the obtained state-space systems are non-minimal and of high dimension, thus requiring a large amount of data to control (see e.g. [8, Section VIC]). In addition, the system matrices

$*$ Reduction of computational complexity:

Recall that the input-state data are informative for quadratic stabilization $\Longleftrightarrow$ there exist $P > 0$, $\beta > 0$ and $K \in \mathbb{R}^{m \times n}$ s.t.

$$\begin{bmatrix} P - \beta I & 0 \\ 0 & -\begin{pmatrix} I \\ K \end{pmatrix} P \begin{pmatrix} I \\ K \end{pmatrix}^T \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^T \geq 0.$$

This inequality could be reformulated as an LMI.

But we can also reduce computational complexity as follows:

$$
\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \hdashline 0 & 0 & K^T \end{bmatrix}^T
\underbrace{\left[\begin{array}{c:c} \begin{bmatrix} P-\beta I & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^T & \begin{matrix} 0 \\ -P \\ 0 \end{matrix} \\ \hdashline \begin{matrix} 0 & \quad -P & \quad 0 \end{matrix} & -P \end{array}\right]}_{=:\,\Psi}
\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \hdashline 0 & 0 & K^T \end{bmatrix} \geqslant 0
$$

So we are looking for
a structured solution
to a QMI!

<u>Idea:</u> Apply corollary with $\Pi = \Psi$, $W = \begin{pmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \end{pmatrix}$.

<u>Theorem:</u> Define $\Theta := \Phi_{12} + X_+ \Phi_{22}$. The data (U_-, X) are informative for quadratic stabilization if and only if there exist $P \in S^n$ and $\beta > 0$ s.t.

$$
\begin{bmatrix} P-\beta I & 0 \\ 0 & -P \end{bmatrix} - \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix} \Phi \begin{bmatrix} I & X_+ \\ 0 & -X_- \end{bmatrix}^T \geqslant 0
$$

$$
P - \beta I - \begin{bmatrix} I \\ X_+^T \end{bmatrix}^T \Phi \begin{bmatrix} I \\ X_+^T \end{bmatrix} + \Theta \begin{bmatrix} X_- \\ U_- \end{bmatrix}^T \left(\begin{bmatrix} X_- \\ U_- \end{bmatrix} \Phi_{22} \begin{bmatrix} X_- \\ U_- \end{bmatrix}^T \right)^{\dagger} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \Theta^T \geqslant 0.
$$

If these LMIs are feasible, then

$$
K := \left(U_- (\Phi_{22} + \Theta^T \Gamma^+ \Theta) X_-^T\right)\left(X_- (\Phi_{22} + \Theta^T \Gamma^+ \Theta) X_-^T\right)^{\dagger}
$$

is a stabilizing feedback gain for all $(A,B) \in \Sigma$, where $\Gamma := P - \beta I - \begin{bmatrix} I & X_+ \end{bmatrix} \Phi \begin{bmatrix} I \\ X_+^T \end{bmatrix}$.

Note: Dimensions of LMIs is smaller, and less decision variables. Explicit formula for controller, given P and β.

Remark: This formulation will also play an important role for getting insight into the conservatism of common Lyapunov functions (see WeBT05.5).

Conclusions

Main message: Results on QMIs as a "toolbox for data-driven robust control"

* Solution sets of QMIs describe
  - sets of systems consistent with data
  - stability/performance guarantees via Lyapunov & dissipation inequalities.

* Inclusion of solution sets $\leftrightarrow$ matrix S-lemmas
  - LMI conditions for stabilization, H_2, H_∞, ...

* Image of solution sets under linear maps
  - "structured" solutions to QMIs
  - input-output systems, noise in subspace, reducing complexity.

QUADRATIC MATRIX INEQUALITIES WITH APPLICATIONS TO DATA-BASED CONTROL*

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Abstract. This paper studies several problems related to quadratic matrix inequalities (QMI's), i.e., inequalities in the Loewner order involving quadratic functions of matrix variables. In particular, we provide conditions under which the solution set of a QMI is nonempty, convex, bounded, or has nonempty interior. We also provide a parameterization of the solution set of a given QMI. In addition, we state results regarding the image of such sets under linear maps, which characterize a subset of "structured" solutions to a QMI. Thereafter, we derive matrix versions of the classical S-lemma and Finsler's lemma,

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and

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