The shortest experiment for linear system identification

Henk van Waarde

Bernoulli Institute for Mathematics, CS and AI
and
Jan C. Willems Center for Systems and Control

University of Groningen

Workshop “Data-driven control: theory and applications”
CDC 2024, Milan

Joint work with Kanat Camlibel and Paolo Rapisarda

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True system:

$$x(t+1) = A_{\mathrm{true}} x(t) + B_{\mathrm{true}} u(t)$$
$$y(t) = C_{\mathrm{true}} x(t) + D_{\mathrm{true}} u(t)$$

$\begin{bmatrix} A_{\mathrm{true}} & B_{\mathrm{true}} \\ C_{\mathrm{true}} & D_{\mathrm{true}} \end{bmatrix} \in \mathbb{R}^{(n_{\mathrm{true}}+p)\times(n_{\mathrm{true}}+m)}$ and n_{true} are **unknown**

Observability matrix and lag:
For $k \geqslant 0$ we define

$$\Omega_k = \begin{cases} 0_{0,n} & \text{if } k = 0 \\ \begin{bmatrix} C_{\mathrm{true}} \\ C_{\mathrm{true}} A_{\mathrm{true}} \\ \vdots \\ C_{\mathrm{true}} A_{\mathrm{true}}^{k-1} \end{bmatrix} & \text{if } k \geqslant 1 \end{cases}$$

The **lag** is defined as the smallest integer $\ell \geqslant 0$ such that $\operatorname{rank} \Omega_\ell = \operatorname{rank} \Omega_{\ell+1}$ and denoted by $\ell_{\mathrm{true}} = \ell(C_{\mathrm{true}}, A_{\mathrm{true}}) \leqslant n_{\mathrm{true}}$.

Experiment design

True system:

$$\begin{aligned} x(t+1) &= A_{\text{true}} x(t) + B_{\text{true}} u(t) \\ y(t) &= C_{\text{true}} x(t) + D_{\text{true}} u(t) \end{aligned} \tag{1}$$

$\begin{bmatrix} A_{\text{true}} & B_{\text{true}} \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix} \in \mathbb{R}^{(n_{\text{true}}+p)\times(n_{\text{true}}+m)}$ and n_{true} are **unknown**

Prior knowledge: (1) is **controllable** and **observable**, $\ell_{\text{true}} \leqslant L$ and $n_{\text{true}} \leqslant N$

Fundamental question: How to find $T \in \mathbb{N}$ and

$$u_{[0,T-1]} := \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix}$$

such that the resulting **data** $(u_{[0,T-1]}, y_{[0,T-1]})$ **enable system identification**?

I.e., such that we can **identify** n_{true} and matrices A, B, C and D satisfying

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{\text{true}} & B_{\text{true}} \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & I \end{bmatrix} \quad \text{for some invertible } S$$

A partial answer

fundamental lemma

A note on persistency of excitation

Jan C. Willems[a], Paolo Rapisarda[b], Ivan Markovsky[a,*], Bart L.M. De Moor[a]

[a]*ESAT, SCD/SISTA, K.U. Leuven, Kasteelpark Arenberg 10, B 3001 Leuven, Heverlee, Belgium*
[b]*Department of Mathematics, University of Maastricht, 6200 MD Maastricht, The Netherlands*

Received 3 June 2004; accepted 7 September 2004
Available online 30 November 2004

Abstract

We prove that if a component of the response signal of a controllable linear time-invariant system is persistently exciting of sufficiently high order, then the windows of the signal span the full system behavior. This is then applied to obtain conditions

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Definition: The input $u_{[0,T-1]}$ is called **persistently exciting** of order k if

$$\operatorname{rank} H_k(u_{[0,T-1]}) = \operatorname{rank} \begin{bmatrix} u(0) & u(1) & \cdots & u(T-k) \\ \vdots & \vdots & & \vdots \\ u(k-1) & u(k) & \cdots & u(T-1) \end{bmatrix} = km$$

A partial answer

fundamental lemma

Possible solution:

- Choose $T := (N+L+1)m + N + L$
- Design $u_{[0,T-1]}$ to be persistently exciting of order $N+L+1$
- Then by the **fundamental lemma**,

$$\operatorname{rank}\begin{bmatrix} H_{L+1}(u_{[0,T-1]}) \\ \hdashline H_{L+1}(y_{[0,T-1]}) \end{bmatrix} = \begin{bmatrix} u(0) & \cdots & u(T-L-1) \\ \vdots & & \vdots \\ u(L) & \cdots & u(T-1) \\ \hdashline y(0) & \cdots & y(T-L-1) \\ \vdots & & \vdots \\ y(L) & \cdots & y(T-1) \end{bmatrix} = (L+1)m + n_{\mathrm{true}}$$

- Apply **subspace identification** to obtain A, B, C and D

We will now consider a simple example...

A partial answer

example

True system and initial state:

$$A_{\mathrm{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \; B_{\mathrm{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_{\mathrm{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D_{\mathrm{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \; x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Hence, $n_{\mathrm{true}} = 2$ and $\ell_{\mathrm{true}} = 1$. We take $N = L = 2$.

Define $T = 9$ and $u_{[0,8]} := \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ (PE of order 5)

$$\operatorname{rank} \begin{bmatrix} H_3(u_{[0,8]}) \\ \hdashline H_3(y_{[0,8]}) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hdashline -1 & 0 & 0 & 1 & 1 & 2 & 3 \\ 2 & 2 & 1 & 1 & 2 & 3 & 7 \\ 0 & 0 & 1 & 1 & 2 & 3 & 5 \\ 2 & 1 & 1 & 2 & 3 & 7 & 9 \\ 0 & 1 & 1 & 2 & 3 & 5 & 9 \\ 1 & 1 & 2 & 3 & 7 & 9 & 14 \end{bmatrix} = 5 \quad \Longrightarrow \quad n_{\mathrm{true}} = 2$$

A partial answer

fundamental lemma

Possible solution:

- Choose $T := (N + L + 1)m + N + L$
- Design $u_{[0,T-1]}$ to be persistently exciting of order $N + L + 1$
- Then by the **fundamental lemma**,

$$\operatorname{rank} \begin{bmatrix} H_{L+1}(u_{[0,T-1]}) \\ H_{L+1}(y_{[0,T-1]}) \end{bmatrix} = \begin{bmatrix} u(0) & \cdots & u(T-L-1) \\ \vdots & & \vdots \\ u(L) & \cdots & u(T-1) \\ y(0) & \cdots & y(T-L-1) \\ \vdots & & \vdots \\ y(L) & \cdots & y(T-1) \end{bmatrix} = (L+1)m + n_{\mathrm{true}}$$

- Apply **subspace identification** to obtain A, B, C and D

Question: Is this the **smallest possible** T? **Answer**: **no!**

Another partial answer

IEEE CONTROL SYSTEMS LETTERS, VOL. 6, 2022 319

Beyond Persistent Excitation: Online Experiment Design for Data-Driven Modeling and Control

Henk J. van Waarde

***Abstract*—This letter presents a new experiment design method for data-driven modeling and control. The idea is to select inputs *online* (using past input/output data), leading to desirable rank properties of data Hankel matrices. In**

rank property is important, since it guarantees that *all* trajectories of the system can be parameterized in terms of the measured trajectory. Essentially, the Hankel matrix of measured inputs and outputs serves as a non-parametric model of

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Possible solution:

- Design the input $u(t)$ **online** based on $(u_{[0,t-1]}, y_{[0,t-1]})$
- T is not specified a priori, but procedure terminates after $T = (L+1)m + n_{\text{true}} + L$ steps
- Apply **subspace identification** to obtain A, B, C and D

We again consider an example...

Another partial answer

example

True system and initial state:

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \; B_{\text{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_{\text{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D_{\text{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \; x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Define $u_{[0,2]} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \neq 0$, measure $y_{[0,2]} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$;

$$\text{rank} \begin{bmatrix} H_3(u_{[0,2]}) \\ \hdashline H_2(y_{[0,1]}) \end{bmatrix} = \text{rank} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \hdashline -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = 1$$

Another partial answer

example

True system and initial state:

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \; B_{\text{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_{\text{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D_{\text{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \; x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Define $u_{[0,2]} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \neq 0$, measure $y_{[0,2]} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$;

$$\operatorname{rank} \begin{bmatrix} H_3(u_{[0,3]}) \\ \hdashline H_2(y_{[0,2]}) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \boldsymbol{u(3)} \\ \hdashline -1 & 0 \\ 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = 2 \text{ for } \boldsymbol{u(3)} = 1$$

Measure $y(3) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Another partial answer

example

True system and initial state:

$$A_{\mathrm{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \; B_{\mathrm{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_{\mathrm{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D_{\mathrm{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \; x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Define $u_{[0,2]} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \neq 0$ and design the rest of the inputs online

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$$\operatorname{rank} \begin{bmatrix} H_3(u_{[0,4]}) \\ \hdashline H_2(y_{[0,3]}) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & u(4) \\ \hdashline -1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 3 \text{ for any } u(4)$$

Take $u(4) = 0$

Another partial answer

example

True system and initial state:

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \; B_{\text{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_{\text{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D_{\text{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \; x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Define $u_{[0,2]} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \neq 0$ and design the rest of the inputs online

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$$\operatorname{rank} \begin{bmatrix} H_3(u_{[0,5]}) \\ \hdashline H_2(y_{[0,4]}) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & u(5) \\ \hdashline -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} = 4 \text{ for any } u(5)$$

Take $u(5) = 0$

Another partial answer

example

True system and initial state:

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},\ B_{\text{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_{\text{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ D_{\text{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},\ x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Define $u_{[0,2]} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \neq 0$ and design the rest of the inputs online

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$$\text{rank} \begin{bmatrix} H_3(u_{[0,6]}) \\ \hdashline H_2(y_{[0,5]}) \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & u(6) \\ \hdashline -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \end{bmatrix} = 5 \text{ for any } u(6)$$

So we take $u(6) = 0$.

Another partial answer

example

True system and initial state:

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \; B_{\text{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_{\text{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D_{\text{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \; x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Define $u_{[0,2]} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \neq 0$ and design the rest of the inputs online

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$$\operatorname{rank} \begin{bmatrix} H_3(u_{[0,7]}) \\ \hdashline H_2(y_{[0,6]}) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & u(7) \\ \hdashline -1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 & 2 \end{bmatrix} = 5 \neq 6 \text{ for any } u(7)$$

So we do not apply $u(7)$ and **stop the procedure**.

Another partial answer

example

True system and initial state:

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \; B_{\text{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_{\text{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D_{\text{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \; x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Define $u_{[0,2]} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \neq 0$ and design the rest of the inputs online

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It follows that

$$\operatorname{rank} \begin{bmatrix} H_3(u_{[0,6]}) \\ \hdashline H_3(y_{[0,6]}) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hdashline -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 2 \end{bmatrix} = (L+1)m + n_{\text{true}} = 5 \implies n_{\text{true}} = 2$$

Reduction # of samples: from $T = 9$ to $T = 7$

Another partial answer

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Beyond Persistent Excitation: Online Experiment Design for Data-Driven Modeling and Control

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rank property is important, since it guarantees that *all* trajectories of the system can be parameterized in terms of the measured trajectory. Essentially, the Hankel matrix of measured inputs and outputs serves as a non-parametric model of

---

Possible solution:

- Design the input $u(t)$ **online** based on $(u_{[0,t-1]}, y_{[0,t-1]})$
- T is not specified a priori, but procedure terminates after $T = (L+1)m + n_{\text{true}} + L$ steps
- Apply **subspace identification** to obtain A, B, C and D

Question: Is this the **smallest possible** T? **Answer**: **it's a secret!**

Informativity for system identification

Informativity for system identification

Beyond the fundamental lemma:
from finite time series to linear system

M. Kanat Camlibel[1] and Paolo Rapisarda[2]

[1]Bernoulli Institute, University of Groningen
[2]School of Electronics and Computer Science, University of Southampton

Abstract

We state necessary and sufficient conditions to uniquely identify (modulo state isomorphism) a linear time-invariant minimal input-state-output system from finite input-output data and upper- and lower bounds on lag and state space dimension.

Data: Let $(u_{[0,T-1]}, y_{[0,T-1]})$ be generated by the true system (no assumptions on the input for now!)

Question: Under what conditions on $(u_{[0,T-1]}, y_{[0,T-1]})$ can we **uniquely identify** the true system (up to state-space transformations)?

System classes

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \quad \longleftrightarrow \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{(n+p)\times(n+m)}$$

$\mathcal{S} = \{ \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathbb{R}^{(n+\boldsymbol{p})\times(n+\boldsymbol{m})} \mid n \geqslant 0 \}$ — systems with $\boldsymbol{m}$ inputs and $\boldsymbol{p}$ outputs

$\mathcal{O} = \{ \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{S} \mid (C, A) \text{ is } \textbf{observable} \}$ — observable systems

$\mathcal{M} = \{ \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{O} \mid (A, B) \text{ is } \textbf{controllable} \}$ — minimal systems

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$\mathcal{S}(\boldsymbol{n}) = \{ \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{S} \mid A \in \mathbb{R}^{\boldsymbol{n}\times\boldsymbol{n}} \}$ — systems with $\boldsymbol{n}$ states

$\mathcal{S}(\boldsymbol{\ell}, \boldsymbol{n}) = \{ \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{S}(n) \mid \ell(C, A) = \boldsymbol{\ell} \}$ — systems with lag $\boldsymbol{\ell}$ and $\boldsymbol{n}$ states

Explaining systems

Definition: A system $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathbb{R}^{(n+p)\times(n+m)}$ explains the data $(u_{[0,T-1]}, y_{[0,T-1]})$ if

$$\begin{bmatrix} \boldsymbol{x_{[1,T]}} \\ y_{[0,T-1]} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \boldsymbol{x_{[0,T-1]}} \\ u_{[0,T-1]} \end{bmatrix}$$

for some $\boldsymbol{x_{[0,T]}} \in \mathbb{R}^{n\times(T+1)}$.

$\boldsymbol{\mathcal{E}} = \left\{ \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{S} \mid \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \text{ explains the data } (u_{[0,T-1]}, y_{[0,T-1]}) \right\}$ explaining systems

$\boldsymbol{\mathcal{E}(n)} = \mathcal{E} \cap \mathcal{S}(n)$ explaining systems with $\boldsymbol{n}$ states

$\boldsymbol{\mathcal{E}(\ell, n)} = \mathcal{E} \cap \mathcal{S}(\ell, n)$ explaining systems with lag $\boldsymbol{\ell}$ and $\boldsymbol{n}$ states

---

True system:

$$\begin{bmatrix} A_{\mathrm{true}} & B_{\mathrm{true}} \\ C_{\mathrm{true}} & D_{\mathrm{true}} \end{bmatrix} \in \mathcal{E}(\ell_{\mathrm{true}}, n_{\mathrm{true}}) \subseteq \mathcal{E}(n_{\mathrm{true}}) \subseteq \mathcal{E}$$

System identification

Prior knowledge: $\mathcal{S}_{\text{pk}} \subseteq \mathcal{S}$ with $\begin{bmatrix} A_{\text{true}} & B_{\text{true}} \\ C_{\text{true}} & D_{\text{true}} \end{bmatrix} \in \mathcal{S}_{\text{pk}}$

Upper bounds on the lag and state dimension:

- Recall that

$$\ell_{\text{true}} \leqslant L \quad \text{and} \quad n_{\text{true}} \leqslant N$$

- Define

$$\mathcal{S}_{L,N} := \left\{ \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{S}(\ell, n) \mid \ell \leqslant L \text{ and } n \leqslant N \right\}$$

- Our prior knowledge is thus:

$$\mathcal{S}_{\text{pk}} = \mathcal{S}_{L,N} \cap \mathcal{M}$$

Definition: The data $(u_{[0,T-1]}, y_{[0,T-1]})$ are informative for SysId if

- $\mathcal{E} \cap \mathcal{S}_{\text{pk}} = \mathcal{E}(n_{\text{true}}) \cap \mathcal{S}_{\text{pk}}$ (data determine state dimension)
- Any pair of systems in $\mathcal{E} \cap \mathcal{S}_{\text{pk}}$ is isomorphic

Necessary and sufficient conditions

$\boldsymbol{\ell}_{\min} = \min\{\ell \geqslant 0 \mid \exists\, n \geqslant 0 \text{ s.t. } \mathcal{E}(\ell, n) \neq \varnothing\}$ minimum lag to explain the data

$\boldsymbol{n}_{\min} = \min\{n \geqslant 0 \mid \mathcal{E}(n) \neq \varnothing\}$ minimum state dimension to explain the data

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Theorem: $\mathcal{E}(\ell, n) \neq \varnothing \implies n - \ell \geqslant n_{\min} - \ell_{\min}$ $\implies \ell \leqslant n - n_{\min} + \ell_{\min}$

Observation: $L_{\mathrm{d}} := N - n_{\min} + \ell_{\min}$ data-guided bound on lag

$L_{\mathrm{a}} := \min(L, L_{\mathrm{d}})$ actual upper bound

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Theorem (Camlibel and Rapisarda, 2024): The data $(u_{[0,T-1]}, y_{[0,T-1]})$ are informative for SysId if and only if

$$T \geqslant L_{\mathrm{a}} + (L_{\mathrm{a}} + 1)m + n_{\min}$$

and

$$\operatorname{rank} \begin{bmatrix} H_{L_{\mathrm{a}}+1}(u_{[0,T-1]}) \\ H_{L_{\mathrm{a}}+1}(y_{[0,T-1]}) \end{bmatrix} = (L_{\mathrm{a}} + 1)m + n_{\min}.$$

Moreover, if these conditions are satisfied then $\ell_{\mathrm{true}} = \ell_{\min}$ and $n_{\mathrm{true}} = n_{\min}$.

The shortest experiment

Necessary conditions for informativity

Recall: $L_\mathrm{a} := \min(L, N - n_\mathrm{min} + \ell_\mathrm{min})$

Theorem: The data $(u_{[0,T-1]}, y_{[0,T-1]})$ are informative for SysId **if and only if**

$$T \geqslant L_\mathrm{a} + (L_\mathrm{a}+1)m + n_\mathrm{min} \quad \text{and} \quad \operatorname{rank}\begin{bmatrix} H_{L_\mathrm{a}+1}(u_{[0,T-1]}) \\ H_{L_\mathrm{a}+1}(y_{[0,T-1]}) \end{bmatrix} = (L_\mathrm{a}+1)m + n_\mathrm{min}.$$

Moreover, if these conditions are satisfied, then $\ell_\mathrm{true} = \ell_\mathrm{min}$ and $n_\mathrm{true} = n_\mathrm{min}$.

Observation: The **shortest** possible informative data length is

$$\boldsymbol{T} := \boldsymbol{L} + (\boldsymbol{L}+1)m + n_\mathrm{true} \quad \text{where} \quad \boldsymbol{L} := \min(L, N - n_\mathrm{true} + \ell_\mathrm{true})$$

Question: Is it possible to **generate** informative data $(u_{[0,\boldsymbol{T}-1]}, y_{[0,\boldsymbol{T}-1]})$, i.e,

$$\operatorname{rank}\begin{bmatrix} H_{\boldsymbol{L}+1}(u_{[0,T-1]}) \\ H_{\boldsymbol{L}+1}(y_{[0,T-1]}) \end{bmatrix} = (\boldsymbol{L}+1)m + n_\mathrm{true}$$

without knowing ℓ_true and n_true?

Preparation

For the data $(u_{[0,t-1]}, y_{[0,t-1]})$, define

$$H_k^t = \begin{bmatrix} u(0) & \cdots & u(t-k) \\ \vdots & & \vdots \\ u(k-1) & \cdots & u(t-1) \\ \hline y(0) & \cdots & y(t-k) \\ \vdots & & \vdots \\ y(k-1) & \cdots & y(t-1) \end{bmatrix}, \qquad G_k^t = \begin{bmatrix} u(0) & \cdots & u(t-k) \\ \vdots & & \vdots \\ u(k-1) & \cdots & u(t-1) \\ \hline y(0) & \cdots & y(t-k) \\ \vdots & & \vdots \\ y(k-2) & \cdots & y(t-2) \end{bmatrix},$$

$$\ell_{\min}^t, \quad n_{\min}^t, \quad \text{and} \quad L_{\mathrm{a}}^t := \min(L, N - n_{\min}^t + \ell_{\min}^t).$$

Main idea: start with $k = 1$ and iterate between the following steps: :

- increase the rank of G_k^t until no progress can be made
- increase the depth k by one

Important question: when to stop?

Stopping criteria

Simple observation: We have that

$$\text{rank}\, \boldsymbol{G}_k^t \leqslant m + \text{rank}\, \boldsymbol{H}_{k-1}^t$$

Lemma: If

$$\text{rank}\, \boldsymbol{G}_k^t < m + \text{rank}\, \boldsymbol{H}_{k-1}^t,$$

then there exists an $m-1$ dimensional affine set $\mathcal{A}^t \subseteq \mathbb{R}^m$ such that

$$\text{rank}\, \boldsymbol{G}_k^{t+1} = \text{rank}\, \boldsymbol{G}_k^t + \mathbf{1} \quad \text{whenever} \quad u(t) \notin \mathcal{A}^t.$$

Theorem: Suppose that $(u_{[0,t-1]}, y_{[0,t-1]})$ is such that

- $\boldsymbol{H}_k^t$ has full column rank, and
- $\text{rank}\, \boldsymbol{G}_k^t = m + \text{rank}\, \boldsymbol{H}_{k-1}^t$.

Then, $k = L_{\mathrm{a}}^t + 1$ implies that

1. $k = \boldsymbol{L} + 1$,
2. $t = \boldsymbol{T}$, and
3. $(u_{[0,\boldsymbol{T}-1]}, y_{[0,\boldsymbol{T}-1]})$ are **informative for SysId**.

The shortest experiment

algorithm

procedure OnlineExperiment(L, N)
  choose $u_{[0,m-1]}$ nonsingular
  measure outputs $y_{[0,m-1]}$
  $t \leftarrow m, \; k \leftarrow 1$
  while $k \neq L_{\mathrm{a}}^{t} + 1$ **do** ▷ stopping criteria
    $k \leftarrow k + 1$
    if $t = k - 1$ **then**
      choose $u(t)$ arbitrarily ▷ $\boldsymbol{G}_k^t$ has (full) rank 1
      measure output $y(t)$
      $t \leftarrow t + 1$
    end if
    while $\operatorname{rank} \boldsymbol{G}_k^t < m + \operatorname{rank} \boldsymbol{H}_{k-1}^t$ **do**
      choose $u(t) \notin \mathcal{A}^t$ ▷ $\operatorname{rank} \boldsymbol{G}_k^{t+1} = \operatorname{rank} \boldsymbol{G}_k^t + 1$
      measure output $y(t)$
      $t \leftarrow t + 1$
    end while
  end while
  return $(u_{[0,t-1]}, y_{[0,t-1]})$ ▷ $(k, t) = (\boldsymbol{L} + 1, \boldsymbol{T})$ and data are informative
end procedure

The shortest experiment

example

True system and initial state:

$$A_{\rm true} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \; B_{\rm true} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_{\rm true} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D_{\rm true} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \; x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Hence, $n_{\rm true} = 2$ and $\ell_{\rm true} = 1$. We take $N = L = 2$.

$u(0) = 1 \implies y(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Let $t = 1$ and $k = 1$.

$n^1_{\rm min} = 0$ and $\ell^1_{\rm min} = 0 \implies L^1_{\rm a} = \min(L, N - n^1_{\rm min} + \ell^1_{\rm min}) = 2 \implies k \neq L^1_{\rm a} + 1$

Set $k = 2$. Since $t = k - 1$, let $u(1) = 0$ (arbitrary) $\implies y(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Now **increase rank**:

$$G^3_2 = \begin{bmatrix} 1 & 0 \\ 0 & u(2) \\ -1 & 0 \\ 2 & 0 \end{bmatrix}$$

The shortest experiment

example

True system and initial state:

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \; B_{\text{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_{\text{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D_{\text{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \; x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Hence, $n_{\text{true}} = 2$ and $\ell_{\text{true}} = 1$. We take $N = L = 2$.

$u(0) = 1 \implies y(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Let $t = 1$ and $k = 1$.

$n^1_{\text{min}} = 0$ and $\ell^1_{\text{min}} = 0 \implies L^1_{\text{a}} = \min(L, N - n^1_{\text{min}} + \ell^1_{\text{min}}) = 2 \implies k \neq L^1_{\text{a}} + 1$

Set $k = 2$. Since $t = k - 1$, let $u(1) = 0$ (arbitrary) $\implies y(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Now **increase rank**:

$$G^3_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 2 & 0 \end{bmatrix}$$

The shortest experiment

example

True system and initial state:

$$A_{\rm true} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_{\rm true} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{\rm true} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{\rm true} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

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$n^1_{\rm min} = 0$ and $\ell^1_{\rm min} = 0 \implies L^1_{\rm a} = \min(L, N - n^1_{\rm min} + \ell^1_{\rm min}) = 2 \implies k \neq L^1_{\rm a} + 1$

Set $k = 2$. Since $t = k - 1$, let $u(1) = 0$ (arbitrary) $\implies y(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Now **increase rank**:

$$G^4_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & u(3) \\ -1 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

The shortest experiment

example

True system and initial state:

$$A_{\mathrm{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \; B_{\mathrm{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_{\mathrm{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D_{\mathrm{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \; x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Hence, $n_{\mathrm{true}} = 2$ and $\ell_{\mathrm{true}} = 1$. We take $N = L = 2$.

$u(0) = 1 \implies y(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Let $t = 1$ and $k = 1$.

$n^1_{\min} = 0$ and $\ell^1_{\min} = 0 \implies L^1_{\mathrm{a}} = \min(L, N - n^1_{\min} + \ell^1_{\min}) = 2 \implies k \neq L^1_{\mathrm{a}} + 1$

Set $k = 2$. Since $t = k - 1$, let $u(1) = 0$ (arbitrary) $\implies y(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Now **increase rank**:

$$G^4_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

The shortest experiment
example

True system and initial state:

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \; B_{\text{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_{\text{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D_{\text{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \; x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Hence, $n_{\text{true}} = 2$ and $\ell_{\text{true}} = 1$. We take $N = L = 2$.

$u(0) = 1 \implies y(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Let $t = 1$ and $k = 1$.

$n^1_{\min} = 0$ and $\ell^1_{\min} = 0 \implies L^1_{\text{a}} = \min(L, N - n^1_{\min} + \ell^1_{\min}) = 2 \implies k \neq L^1_{\text{a}} + 1$

Set $k = 2$. Since $t = k - 1$, let $u(1) = 0$ (arbitrary) $\implies y(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Now **increase rank**:

$$G^5_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & u(4) \\ -1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 \end{bmatrix}$$

The shortest experiment

example

True system and initial state:

$$A_{\rm true} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \; B_{\rm true} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_{\rm true} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D_{\rm true} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \; x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Hence, $n_{\rm true} = 2$ and $\ell_{\rm true} = 1$. We take $N = L = 2$.

$u(0) = 1 \implies y(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Let $t = 1$ and $k = 1$.

$n^1_{\rm min} = 0$ and $\ell^1_{\rm min} = 0 \implies L^1_{\rm a} = \min(L, N - n^1_{\rm min} + \ell^1_{\rm min}) = 2 \implies k \neq L^1_{\rm a} + 1$

Set $k = 2$. Since $t = k - 1$, let $u(1) = 0$ (arbitrary) $\implies y(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Now **increase rank**:

$$G^5_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 \end{bmatrix}$$

The shortest experiment

example

True system and initial state:

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \ B_{\text{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ C_{\text{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ D_{\text{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \ x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Hence, $n_{\text{true}} = 2$ and $\ell_{\text{true}} = 1$. We take $N = L = 2$.

$$\text{rank}\, H_1^5 = \text{rank} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 2 & 1 & 1 \end{bmatrix} = 3 \implies \boxed{\text{rank}\, G_2^5 = 1 + \text{rank}\, H_1^5}$$

$$\ell_{\min}^5 = 1 \text{ and } n_{\min}^5 = 2 \implies L_{\text{a}}^5 = \min(2, 2-2+1) = 1 \implies k = L_{\text{a}}^5 + 1.$$

Conclusion: The data $(u_{[0,4]}, y_{[0,4]})$ are **informative for SysId**

Reduction in # samples: from $T = 9$ to $T = 7$ to $\boldsymbol{T} = 5$

Conclusions

The shortest experiments for system identification require:

1. **Online design** of the inputs
2. **Online adaptation** of the **depth** of the Hankel matrix

---

Online design using depth-$(L+1)$ Hankel matrix is shortest only if

$$L = N - n_{\text{true}} + \ell_{\text{true}}$$

Final example: For a system with

$$m = 80, \quad p = 10, \quad \ell_{\text{true}} = 20, \quad n_{\text{true}} = 100,$$

and

$$L = 100, \quad N = 150,$$

- fundamental lemma requires: $T = 20330$
- online design (fixed depth) requires: $T = 8280$
- the shortest experiment requires: $\boldsymbol{T} = 5850$

Thank you!

Minimum lag and state dimension

$\boldsymbol{\ell}_{\min} = \min\{\ell \geqslant 0 \mid \exists\, n \geqslant 0 \text{ s.t. } \mathcal{E}(\ell, n) \neq \varnothing\}$ minimum lag to explain the data

$\boldsymbol{n}_{\min} = \min\{n \geqslant 0 \mid \mathcal{E}(n) \neq \varnothing\}$ minimum state dimension to explain the data

---

Question: How can we obtain $\ell_{\min}$ and $n_{\min}$ from the data?

Important role played by the **Hankel matrices**:

$$H_{\boldsymbol{k}} = \begin{bmatrix} u(0) & \cdots & u(T-k) \\ \vdots & & \vdots \\ u(\boldsymbol{k-1}) & \cdots & u(T-1) \\ \hdashline y(0) & \cdots & y(T-k) \\ \vdots & & \vdots \\ y(\boldsymbol{k-1}) & \cdots & y(T-1) \end{bmatrix} \quad \text{and} \quad G_{\boldsymbol{k}} = \begin{bmatrix} u(0) & \cdots & u(T-k) \\ \vdots & & \vdots \\ u(\boldsymbol{k-1}) & \cdots & u(T-1) \\ \hdashline y(0) & \cdots & y(T-k) \\ \vdots & & \vdots \\ y(\boldsymbol{k-2}) & \cdots & y(T-2) \end{bmatrix}$$

Minimum lag and state dimension

$\ell_{\min} = \min\{\ell \geqslant 0 \mid \exists\, n \geqslant 0 \text{ s.t. } \mathcal{E}(\ell, n) \neq \varnothing\}$ minimum lag to explain the data

$n_{\min} = \min\{n \geqslant 0 \mid \mathcal{E}(n) \neq \varnothing\}$ minimum state dimension to explain the data

---

Assumption: $u_{[0,T-1]} \neq 0_{m,T}$ (necessary for SysId)

We define for $k \in [0, T-1]$:

$$\delta_k = \operatorname{rank} H_{k+1} - \operatorname{rank} G_{k+1}$$

Then $p \geqslant \delta_0 \geqslant \cdots \geqslant \delta_{T-1} = 0$

$q :=$ the smallest integer such that δ_q is zero. $q \in [0, T-1]$

Theorem: $\ell_{\min} = q$ and $n_{\min} = \sum_{i=0}^{\ell_{\min}} \delta_i$.

Theorem: $\mathcal{E}(\ell, n) \neq \varnothing \implies n - \ell \geqslant n_{\min} - \ell_{\min}$.