Experiment design
for data-driven modeling, analysis and control

DISC Summer School 2025

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Main question: how to **generate** informative data?

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These lectures: Experiment design for:

- trajectory parameterization
- system identification
- stabilization

Fundamental lemma

Universal inputs

Online experiment design

Experiment design for identification

Experiment design for stabilization

Conclusions

Fundamental lemma

- The fundamental lemma was proven in (Willems et al., 2005).

A note on persistency of excitation

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Abstract

We prove that if a component of the response signal of a controllable linear time-invariant system is persistently exciting of sufficiently high order, then the windows of the signal span the full system behavior. This is then applied to obtain conditions

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- Applications:
  1. System identification via subspace methods (Van Overschee & De Moor, 1996), (Verhaegen & Verdult, 2007).

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     - stabilization (De Persis and Tesi, 2019), and
     - predictive control (Coulson et al., 2019), (Berberich et al., 2020).
- Extensions:
  - multiple datasets (van Waarde et al., 2020),
  - parameter-varying systems (Verhoek et al., 2021)
  - descriptor systems (Schmitz et al., 2022)
  - robust version (Coulson et al., 2022)
  - stochastic systems (Pan et al., 2022)
  - switched systems (Petreczky and Bako, 2023)
  - continuous-time systems (Rapisarda et al., 2023)
  - frequency domain counterpart (Meijer et al., 2023)
  - 2D systems (Rapisarda and Zhang, 2024)
  - and several classes of nonlinear systems.

Consider the input-state-output system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{aligned} \qquad (\star)$$

where $u(t) \in \mathbb{R}^m$, $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$ for all $t \in \mathbb{Z}_+$.

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Identify $(\star)$ with the quadruple $(A, B, C, D) \in \mathcal{M}^{m,n,p}$, where

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Definition: We define the (input-output) behavior of $(A, B, C, D) \in \mathcal{M}^{m,n,p}$ by

$$\mathfrak{B}(A, B, C, D) := \{(u, y) : \mathbb{Z}_+ \to \mathbb{R}^m \times \mathbb{R}^p \mid \exists x : \mathbb{Z}_+ \to \mathbb{R}^n \text{ such that } (\star) \text{ holds for all } t \in \mathbb{Z}_+\}.$$

Given $f : \mathbb{Z}_+ \to \mathbb{R}^q$ and a positive $k \in \mathbb{Z}_+$, we define

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$$H_k(f_{[0,T-1]}) := \begin{bmatrix} f(0) & f(1) & \cdots & f(T-k) \\ f(1) & f(2) & \cdots & f(T-k+1) \\ \vdots & \vdots & & \vdots \\ f(k-1) & f(k) & \cdots & f(T-1) \end{bmatrix}.$$

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$$\mathfrak{B}_k(A, B, C, D) := \left\{ \begin{bmatrix} u_{[0,k-1]} \\ y_{[0,k-1]} \end{bmatrix} \mid (u, y) \in \mathfrak{B}(A, B, C, D) \right\}.$$

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Moreover, we use the shorthand notation: $\mathfrak{B}_k(A, B) := \mathfrak{B}_k(A, B, I_n, 0)$.

Now, consider the **data**: $\begin{bmatrix} u_{[0,T-1]} \\ y_{[0,T-1]} \end{bmatrix} \in \mathfrak{B}_{[0,T-1]}(A, B, C, D).$

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Let $L \in [1,T]$. By time-invariance, each column of

$$\left[\begin{array}{c} H_L(u_{[0,T-1]}) \\ \hdashline H_L(y_{[0,T-1]}) \end{array}\right] = \left[\begin{array}{cccc} u(0) & u(1) & \cdots & u(T-L) \\ \vdots & \vdots & & \vdots \\ u(L-1) & u(L) & \cdots & u(T-1) \\ \hdashline y(0) & y(1) & \cdots & y(T-L) \\ \vdots & \vdots & & \vdots \\ y(L-1) & y(L) & \cdots & y(T-1) \end{array}\right] \qquad (\diamond)$$

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Since $\mathfrak{B}_{[0,L-1]}(A,B,C,D)$ is a subspace, **all linear combinations of the columns of** $(\diamond)$ are also in $\mathfrak{B}_{[0,L-1]}(A,B,C,D)$.

In other words:

$$\operatorname{im}\begin{bmatrix} H_L(u_{[0,T-1]}) \\ H_L(y_{[0,T-1]}) \end{bmatrix} \subseteq \mathfrak{B}_{[0,L-1]}(A,B,C,D).$$

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Important question: Under which conditions do we have

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This would allow us to **parameterize** all length-L trajectories using **data**:

$$\begin{bmatrix} \bar{u}_{[0,L-1]} \\ \bar{y}_{[0,L-1]} \end{bmatrix} \in \mathfrak{B}_{[0,L-1]}(A,B,C,D) \iff \begin{bmatrix} \bar{u}_{[0,L-1]} \\ \bar{y}_{[0,L-1]} \end{bmatrix} = \begin{bmatrix} H_L(u_{[0,T-1]}) \\ H_L(y_{[0,T-1]}) \end{bmatrix} g$$

for some $g \in \mathbb{R}^{T-L+1}$.

Theorem (Willems et al., 2005): Assume that $(A, B, C, D) \in \mathcal{M}^{m,n,p}_{\text{cont}}$ and $u_{[0,T-1]}$ is **persistently exciting** of order $n + L$. Then:

- the rank condition

$$\operatorname{rank} \begin{bmatrix} H_1(x_{[0,T-L]}) \\ H_L(u_{[0,T-1]}) \end{bmatrix} = n + mL$$

  holds for all $x_{[0,T-L]}$ such that $\begin{bmatrix} u_{[0,T-L]} \\ x_{[0,T-L]} \end{bmatrix} \in \mathfrak{B}_{T-L+1}(A, B)$.

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- If only upper bound $N \geqslant n$ is given, use $u_{[0,T-1]}$ that is **PE of order** $N+L$
- If $L > \ell(C,A)$ where $\ell(C,A)$ is the smallest integer such that
$$\operatorname{rank}\begin{bmatrix} C^\top & (CA)^\top & \cdots & (CA^{\ell-1})^\top \end{bmatrix} = \operatorname{rank}\begin{bmatrix} C^\top & (CA)^\top & \cdots & (CA^{\ell})^\top \end{bmatrix}$$
(i.e., the **lag** of the system), then $\mathfrak{B}_{[0,L-1]}$ uniquely determines $\mathfrak{B}$

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A New Perspective on Willems' Fundamental Lemma: Universality of Persistently Exciting Inputs

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$$\begin{bmatrix} H_1(u_{[0,2]}) \\ H_1(y_{[0,2]}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x_1(0) & x_2(0) & 1 \end{bmatrix},$$

which has rank 2 for all $x(0) \in \mathbb{R}^2$.

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which has rank 2 for all $x(0) \in \mathbb{R}^2$. Thus, $\mathfrak{B}_1(A, B, C, D) = \operatorname{im} \begin{bmatrix} H_1(u_{[0,2]}) \\ H_1(y_{[0,2]}) \end{bmatrix}$.

Conclusion: For a **single** system, persistency of excitation is **not necessary**.

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2. But are there other universal inputs?

A New Perspective on Willems' Fundamental Lemma: Universality of Persistently Exciting Inputs

Amir Shakouri, Henk J. van Waarde, M. Kanat Camlibel

***Abstract*— In this letter, we provide new insight into Willems et al.'s fundamental lemma by studying the concept of universal inputs. An input is called *universal* if, when applied to *any* con-**

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Theorem: The input $u_{[0,T-1]}$ is universal for determining the L-restricted behavior **if and only if** it is persistently exciting of order $n+L$.

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Comments:

- The "if" part follows from Willems et al.'s fundamental lemma.

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Theorem: The input $u_{[0,T-1]}$ is universal for determining the L-restricted behavior **if and only if** it is persistently exciting of order $n+L$.

Comments:

- The "if" part follows from Willems et al.'s fundamental lemma.
- For the "only if" part, given an input that is **not** PE, we show how to find
  1. a controllable system and
  2. an initial state

  such that the data **do not** parameterize the L-restricted behavior.

Idea of the proof ("only if"):

1. Suppose that $u_{[0,T-1]}$ is not PE of order $n+L$.
2. Let $\eta \in \ker H_{n+L}(u_{[0,T-1]})^\top$ be a nonzero vector.
3. Partition $\eta^\top = \begin{bmatrix} \eta_0^\top & \cdots & \eta_{n+L-1}^\top \end{bmatrix}$, where $\eta_0, \ldots, \eta_{n+L-1} \in \mathbb{R}^m$.
4. Take $A \in \mathbb{R}^{n\times n}$ and $\zeta \in \mathbb{R}^n$ such that
   - (A, ζ) is controllable, and
   - if λ is an eigenvalue of A then $\sum_{i=0}^{n+L-1} \lambda^i \eta_i \neq 0$.
5. Define $E_{n+L-1} := 0$ and $E_{i-1} := AE_i + \zeta\eta_i^\top$ for $i \in [-1, n+L-1]$.
6. Construct $B := E_{-1}$ and $x(0) := -\sum_{i=0}^{n+L-2} E_i u(i)$. Then (A,B) is controllable.
7. It can be shown that

$$\begin{bmatrix} w^\top & v^\top \end{bmatrix} \begin{bmatrix} H_1(x_{[0,T-L]}) \\ H_L(u_{[0,T-1]}) \end{bmatrix} = 0$$

   for some nonzero $w \in \mathbb{R}^n$ and $v \in \mathbb{R}^{mL}$.
8. Define $C \in \mathbb{R}^{p\times n}$ such that its first row is $w^\top$, and $D := 0$.
9. Finally, it can be shown that $\mathfrak{B}_L(A,B,C,D) \neq \operatorname{im} \begin{bmatrix} H_L(u_{[0,T-1]}) \\ H_L(y_{[0,T-1]}) \end{bmatrix}$.

Conclusion: Universal inputs are precisely the persistently exciting ones.

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Next (natural) question: Can we improve over PE if we care about only the restricted behavior of the **true data-generating system**?

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Next (natural) question: Can we improve over PE if we care about only the restricted behavior of the **true data-generating system**?

Not so obvious, because that system is **not known beforehand**...

Fundamental lemma

Universal inputs

Online experiment design

Experiment design for identification

Experiment design for stabilization

Conclusions

Let's first talk about what is **known** and **unknown**.

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$$(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}}) \in \mathcal{M}_{\text{min}}^{m,n_{\text{true}},p},$$

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Let ℓ_{true} denote the **lag** of $(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})$. Given: $L > \ell_{\text{true}}$.

Problem: How to design $u_{[0,T-1]}$ such that, for a fixed **unknown** $x(0)$,

$$\operatorname{im} \begin{bmatrix} H_L(u_{[0,T-1]}) \\ H_L(y_{[0,T-1]}) \end{bmatrix} = \mathfrak{B}_L(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}}),$$

where $y_{[0,T-1]}$ is the output of the true system resulting from $u_{[0,T-1]}$ and $x(0)$**?**

A partial answer (using fundamental lemma):

1. Since $L > \ell_{\text{true}}$ and $(C_{\text{true}}, A_{\text{true}})$ is observable, $n_{\text{true}} \leqslant (L-1)p =: N$

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3. Then $u_{[0,T-1]}$ is also persistently exciting of order $n_{\text{true}} + L$
4. By the fundamental lemma,

$$\operatorname{im} \begin{bmatrix} H_L(u_{[0,T-1]}) \\ H_L(y_{[0,T-1]}) \end{bmatrix} = \mathfrak{B}_L(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}}).$$

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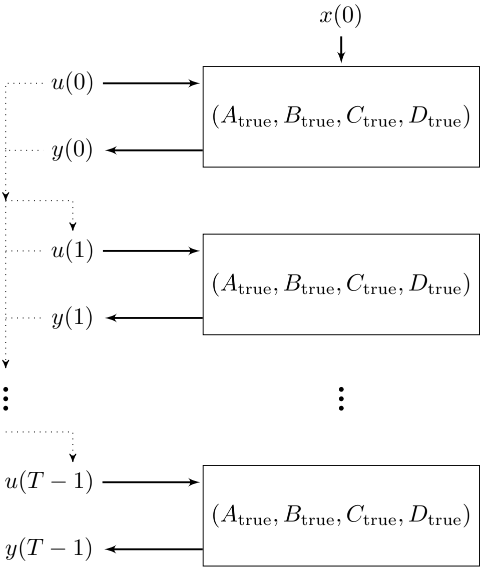
Remarks:

- This requires $T \geqslant (m+1)(N+L) - 1$ samples
- However, as PE inputs are **universal** this may be overkill...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Theorem: Define $T := n_{\text{true}} + (m+1)L - 1$. Let $u_{[0,L-1]}$ be nonzero. For all $t \in [L, T-1]$ there exists an $(m-1)$-dimensional affine set $\mathcal{A}^t \subseteq \mathbb{R}^m$ such that

$$\operatorname{rank}\begin{bmatrix} H_L(u_{[0,t]}) \\ H_{L-1}(y_{[0,t-1]}) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} H_L(u_{[0,t-1]}) \\ H_{L-1}(y_{[0,t-2]}) \end{bmatrix} + \mathbf{1}$$

whenever $u(t) \notin \mathcal{A}^t$.

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Interpretation: We can increase the rank of the Hankel matrix at every time step!

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Corollary: If $u(t) \notin \mathcal{A}^t$ for all $t \in [L, T-1]$ then

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This implies that $\operatorname{im} \begin{bmatrix} H_L(u_{[0,T-1]}) \\ H_L(y_{[0,T-1]}) \end{bmatrix} = \mathfrak{B}_L(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}})$.

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Note: T is **not known** before the experiment.

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Note: T is **not known** before the experiment. It is found online because it equals the smallest t for which

$$\operatorname{rank}\begin{bmatrix} H_L(u_{[0,t]}) \\ H_{L-1}(y_{[0,t-1]}) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} H_L(u_{[0,t-1]}) \\ H_{L-1}(y_{[0,t-2]}) \end{bmatrix}$$

for all $u(t) \in \mathbb{R}^m$.

True system and initial state:

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \; B_{\text{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_{\text{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D_{\text{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \; x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

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Define $L = 3$, $u_{[0,2]} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top \neq 0$, and measure $H_1(y_{[0,2]}) = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$;

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$$\operatorname{rank} \begin{bmatrix} H_3(u_{[0,2]}) \\ \hdashline H_2(y_{[0,1]}) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \hdashline -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = 1$$

True system and initial state:

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Define $u_{[0,2]} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top \neq 0$, and design the rest of the inputs online

---

$$\operatorname{rank} \begin{bmatrix} H_3(u_{[0,3]}) \\ \hdashline H_2(y_{[0,2]}) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \boldsymbol{u(3)} \\ \hdashline -1 & 0 \\ 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = 2 \text{ for } \boldsymbol{u(3)} = 1$$

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Measure $y(3) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

True system and initial state:

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_{\text{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{\text{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{\text{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

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$$\operatorname{rank} \begin{bmatrix} H_3(u_{[0,4]}) \\ \hdashline H_2(y_{[0,3]}) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & u(4) \\ \hdashline -1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 3 \text{ for any } u(4)$$

Take $u(4) = 0$

True system and initial state:

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_{\text{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{\text{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{\text{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Define $u_{[0,2]} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top \neq 0$ and design the rest of the inputs online

---

$$\operatorname{rank} \begin{bmatrix} H_3(u_{[0,5]}) \\ \hdashline H_2(y_{[0,4]}) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & u(5) \\ \hdashline -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} = 4 \text{ for any } u(5)$$

Take $u(5) = 0$

True system and initial state:

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_{\text{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{\text{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{\text{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Define $u_{[0,2]} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top \neq 0$ and design the rest of the inputs online

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$$\operatorname{rank} \begin{bmatrix} H_3(u_{[0,6]}) \\ \hdashline H_2(y_{[0,5]}) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & u(6) \\ \hdashline -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \end{bmatrix} = 5 \text{ for any } u(6)$$

So we take $u(6) = 0$.

True system and initial state:

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \ B_{\text{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ C_{\text{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ D_{\text{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \ x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Define $u_{[0,2]} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top \neq 0$ and design the rest of the inputs online

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$$\operatorname{rank} \begin{bmatrix} H_3(u_{[0,7]}) \\ \hdashline H_2(y_{[0,6]}) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & u(7) \\ \hdashline -1 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 & 2 \end{bmatrix} = 5 \neq 6 \text{ for any } u(7)$$

True system and initial state:

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_{\text{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{\text{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{\text{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

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So we do not apply $u(7)$ and **stop the procedure**.

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$$A_{\text{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \; B_{\text{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_{\text{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D_{\text{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \; x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

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# of samples: $T = 7$ instead of $T \geqslant 13$ required for PE of order $7 = 4 + 3$.

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- Online experiment design tailors $u_{[0,T-1]}$ to the data-generating system
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- The online approach only requires $T = n_{\text{true}} + (m + 1)L - 1$ samples.

Fundamental lemma

Universal inputs

Online experiment design

Experiment design for identification

Experiment design for stabilization

Conclusions

Recall: We have seen two methods that guarantee

$$\operatorname{im}\begin{bmatrix} H_L(u_{[0,T-1]}) \\ H_L(y_{[0,T-1]}) \end{bmatrix} = \mathfrak{B}_L(A_{\text{true}}, B_{\text{true}}, C_{\text{true}}, D_{\text{true}}).$$

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Question: Is this **sample-efficient** or is there a better approach?

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To answer that question, we need conditions on given data $(u_{[0,T-1]}, y_{[0,T-1]})$ that **enable system identification**.

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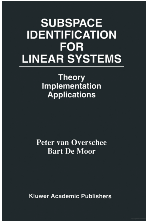

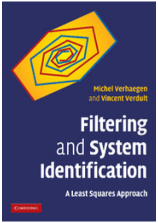

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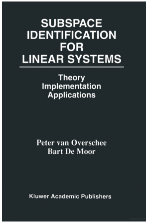

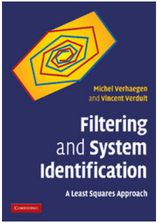

Necessary and sufficient conditions:

Beyond the fundamental lemma:
from finite time series to linear system

M. Kanat Camlibel[1] and Paolo Rapisarda[2]

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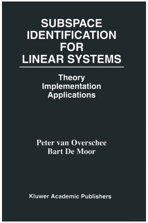

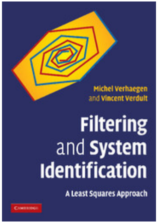

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We will now review these conditions...

Prior knowledge: $\ell_{\text{true}} < L$, $n_{\text{true}} \leqslant N$ and

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1. If only L is given, choose $N := (L-1)p$.
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Definition: The data $(u_{[0,T-1]}, y_{[0,T-1]})$ are **informative for SysId** if

$$\begin{bmatrix} u_{[0,T-1]} \\ y_{[0,T-1]} \end{bmatrix} \in \mathfrak{B}_T(A,B,C,D) \qquad (\triangle)$$

for some $(A,B,C,D) \in \mathcal{M}_{\mathrm{min}}^{m,n,p}$ with $\ell(C,A) < L$ and $n \leqslant N$ implies

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Two important integers:

ℓ_{min} minimum lag of all data-consistent systems

n_{min} minimum state dimension of all data-consistent systems

Fact: $\ell_{\text{true}} < L_{\text{d}} := N - n_{\min} + \ell_{\min} + 1$ data-guided bound on lag

$L_{\text{a}} := \min(L, L_{\text{d}})$ actual upper bound

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$$T \geqslant n_{\min} + (m+1)L_{\text{a}} - 1 \quad \text{and} \quad \operatorname{rank} \begin{bmatrix} H_{L_{\text{a}}}(u_{[0,T-1]}) \\ H_{L_{\text{a}}}(y_{[0,T-1]}) \end{bmatrix} = n_{\min} + mL_{\text{a}}.$$

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Question: Is it possible to **generate** informative data $(u_{[0,\boldsymbol{T}-1]}, y_{[0,\boldsymbol{T}-1]})$, i.e,

$$\operatorname{rank} \begin{bmatrix} H_{\boldsymbol{L}}(u_{[0,\boldsymbol{T}-1]}) \\ H_{\boldsymbol{L}}(y_{[0,\boldsymbol{T}-1]}) \end{bmatrix} = n_{\text{true}} + m\boldsymbol{L}$$

without knowing ℓ_{true} and n_{true} in advance?

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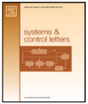

Check for updates

The shortest experiment for linear system identification

M.K. Camlibel [a], H.J. van Waarde [a,*,1], P. Rapisarda [b]

[a] Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen, The Netherlands
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For the data $(u_{[0,t-1]}, y_{[0,t-1]})$, define

$$H_k^t := \begin{bmatrix} H_k(u_{[0,t-1]}) \\ \hdashline H_k(y_{[0,t-1]}) \end{bmatrix}, \qquad G_k^t := \begin{bmatrix} H_k(u_{[0,t-1]}) \\ \hdashline H_{k-1}(y_{[0,t-2]}) \end{bmatrix},$$

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Main idea: start with $k = 1$ and iterate between the following steps:

- increase the rank of G_k^t until no progress can be made

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The shortest experiment for linear system identification

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For the data $(u_{[0,t-1]}, y_{[0,t-1]})$, define

$$H_k^t := \begin{bmatrix} H_k(u_{[0,t-1]}) \\ \hdashline H_k(y_{[0,t-1]}) \end{bmatrix}, \qquad G_k^t := \begin{bmatrix} H_k(u_{[0,t-1]}) \\ \hdashline H_{k-1}(y_{[0,t-2]}) \end{bmatrix},$$

$$\ell_{\min}^t, \quad n_{\min}^t, \quad L_{\mathrm{d}}^t, \quad \text{and} \quad L_{\mathrm{a}}^t := \min(L, L_{\mathrm{d}}^t).$$

Main idea: start with $k = 1$ and iterate between the following steps:

- increase the rank of G_k^t until no progress can be made
- increase the depth k by one

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Main idea: start with $k = 1$ and iterate between the following steps:

- increase the rank of G_k^t until no progress can be made
- increase the depth k by one

Important question: when to stop?

Lemma: We have that

$$\operatorname{rank} \boldsymbol{G}_k^t \leqslant m + \operatorname{rank} \boldsymbol{H}_{k-1}^t$$

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Lemma: If

$$\operatorname{rank} \boldsymbol{G}_k^t < m + \operatorname{rank} \boldsymbol{H}_{k-1}^t,$$

then there exists an $m-1$ dimensional affine set $\mathcal{A}^t \subseteq \mathbb{R}^m$ such that

$$\operatorname{rank} \boldsymbol{G}_k^{t+1} = \operatorname{rank} \boldsymbol{G}_k^t + \mathbf{1} \quad \text{whenever} \quad u(t) \notin \mathcal{A}^t.$$

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Theorem: Suppose that $(u_{[0,t-1]}, y_{[0,t-1]})$ is such that

- $\boldsymbol{H}_k^t$ has full column rank, and
- $\operatorname{rank} \boldsymbol{G}_k^t = m + \operatorname{rank} \boldsymbol{H}_{k-1}^t$.

Then, $k = L_{\mathrm{a}}^t$ implies that

1. $k = \boldsymbol{L}$,
2. $t = \boldsymbol{T}$, and
3. $(u_{[0,\boldsymbol{T}-1]}, y_{[0,\boldsymbol{T}-1]})$ are **informative for SysId**.

Algorithm for shortest experiment

procedure ONLINEEXPERIMENT(L, N)
    choose $u_{[0,m-1]}$ nonsingular
    measure outputs $y_{[0,m-1]}$
    $t \leftarrow m, \;\; k \leftarrow 1$
    while $k \neq L_{\mathrm{a}}^{\boldsymbol{t}}$ **do** ▷ stopping criterion
        $k \leftarrow k + 1$
        if $t = k - 1$ **then**
            choose $u(t)$ arbitrarily ▷ $\boldsymbol{G}_{\boldsymbol{k}}^{\boldsymbol{t}}$ has (full) rank 1
            measure output $y(t)$
            $t \leftarrow t + 1$
        end if
        while $\operatorname{rank} \boldsymbol{G}_{\boldsymbol{k}}^{\boldsymbol{t}} < m + \operatorname{rank} \boldsymbol{H}_{\boldsymbol{k-1}}^{\boldsymbol{t}}$ **do**
            choose $u(t) \notin \mathcal{A}^t$ ▷ $\operatorname{rank} \boldsymbol{G}_{\boldsymbol{k}}^{\boldsymbol{t+1}} = \operatorname{rank} \boldsymbol{G}_{\boldsymbol{k}}^{\boldsymbol{t}} + 1$
            measure output $y(t)$
            $t \leftarrow t + 1$
        end while
    end while
    return $(u_{[0,t-1]}, y_{[0,t-1]})$ ▷ $(k, t) = (\boldsymbol{L}, \boldsymbol{T})$ and data are informative
end procedure

True system and initial state:

$$A_{\text{true}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \ B_{\text{true}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ C_{\text{true}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ D_{\text{true}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \ x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

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$$n_{\min}^1 = 0,\ \ell_{\min}^1 = 0 \implies L_{\text{a}}^1 = \min(L, N - n_{\min}^1 + \ell_{\min}^1 + 1) = 3$$

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Now **increase rank**:

$$G^3_2 = \begin{bmatrix} 1 & 0 \\ 0 & u(2) \\ -1 & 0 \\ 2 & 0 \end{bmatrix}$$

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$\ell_{\rm min}^5 = 1$ and $n_{\rm min}^5 = 2$

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Reduction in # samples for identification: from $T = 7$ to $\boldsymbol{T} = 5$

Conclusion: The shortest experiments for system identification require:

1. **Online design** of the inputs
2. **Online adaptation** of the **depth** of the Hankel matrix

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Larger example: For a system with

$$m = 80, \quad p = 10, \quad \ell_{\text{true}} = 20, \quad n_{\text{true}} = 100,$$

and

$$L = 101, \quad N = 150,$$

- fundamental lemma (PE of order $N + L$) requires: $T = 20330$

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- online design (fixed depth) requires: $T = 8280$

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- fundamental lemma (PE of order $N + L$) requires: $T = 20330$
- online design (fixed depth) requires: $T = 8280$
- the shortest experiment requires: $\boldsymbol{T} = 5850$

Fundamental lemma

Universal inputs

Online experiment design

Experiment design for identification

Experiment design for stabilization

Conclusions

Consider the stabilizable input-state system

$$x(t+1) = A_{\text{true}}x(t) + B_{\text{true}}u(t)$$

where $u(t) \in \mathbb{R}^m$ and $x(t) \in \mathbb{R}^n$ for all $t \in \mathbb{Z}_+$.

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We identify the system with the pair $(A_{\mathrm{true}}, B_{\mathrm{true}}) \in \mathcal{M}^{m,n}$, where

$$\mathcal{M}^{m,n} := \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}.$$

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Definition: The set $\Sigma_{\mathcal{D}}$ of all data-consistent systems is defined as

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Aim: Use the data $\mathcal{D} = (u_{[0,T]}, x_{[0,T]})$ to find a stabilizing feedback gain $K \in \mathbb{R}^{m \times n}$ such that $A_{\text{true}} + B_{\text{true}}K$ is Schur.

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Note: Willems' fundamental lemma **does not apply** (no controllability)...

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Theorem[2]: If $u_{[0,T-1]}$ is **persistently exciting** of order $n+1$ then

$$\operatorname{im} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = (\mathcal{R} + \mathcal{K}) \times \mathbb{R}^m,$$

where

$$\begin{aligned} \mathcal{R} &:= \operatorname{im} \begin{bmatrix} B_{\text{true}} & A_{\text{true}} B_{\text{true}} & \cdots & A_{\text{true}}^{n-1} B_{\text{true}} \end{bmatrix} \\ \mathcal{K} &:= \operatorname{im} \begin{bmatrix} x(0) & A_{\text{true}} x(0) & \cdots & A_{\text{true}}^{n-1} x(0) \end{bmatrix}. \end{aligned}$$

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[2] Yu *et al.*, On controllability and persistency of excitation in data-driven control: Extensions of Willems' fundamental lemma, CDC, 2021.

Theorem[3] If $u_{[0,T-1]}$ is **persistently exciting** of order $n+1$ then $(u_{[0,T]}, x_{[0,T]})$ are informative for stabilization with respect to $\mathcal{M}^{m,n}_{\mathsf{stab}}$ for any $x_{[0,T]}$ such that

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Fundamental lemma

Universal inputs

Online experiment design

Experiment design for identification

Experiment design for stabilization

Conclusions

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Thank you!