Energy-based learning

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Based on joint work with: Anne-Men Huijzer, Tom Chaffey and Bart Besselink

Problem: digital computers consume enormous amounts of energy

- ChatGPT: ± 3 Wh per query, ± 3 GWh per day

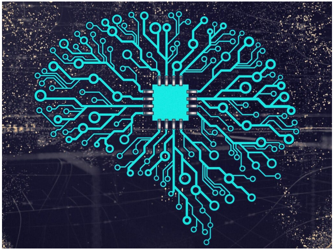
Alternative: analog computing

- discrete values $\rightarrow$ analog signals (voltages/currents)

Prominent example: neuromorphic computing

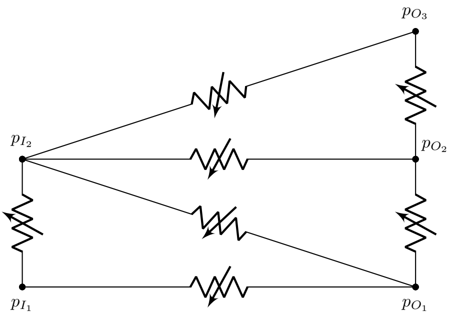
- aim: create circuit elements that behave like biological neurons

"the brain is a factor of 1 billion more efficient than our present digital technology"
–Carver Mead[1]



[1] C. Mead, *Neuromorphic electronic systems*, Proc. IEEE, 78(10): pp 1629–1636, 1990.

This talk: learning from input-output data in resistive electrical circuits



The data: a pair of voltage potentials (p_I, p_O^D)

Goal: Adjust the conductances of the resistors so that the circuit maps p_I to p_O^D

- **Tool**: **Energy-based learning algorithms**

Problem formulation

Graph theory:

- $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ connected undirected graph
- $\mathcal{V} = \{1, 2, \ldots, N\}$ set of nodes
- $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ set of B branches
- $D \in \mathbb{R}^{N \times B}$ incidence matrix

Circuit theory:

- $p \in \mathbb{R}^N$ vector of voltage potentials at nodes
- $j \in \mathbb{R}^N$ nodal currents entering each node
- $v \in \mathbb{R}^B$ voltages across the branches
- $g \in \mathbb{R}^B$ vector of positive conductances
- $G := \mathrm{diag}(g) \in \mathbb{R}^{B \times B}$ diagonal matrix of conductances

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Using the laws of Kirchhoff and Ohm, we get:

$$j = DGD^\top p$$

where $DGD^\top$ is the Laplacian matrix of $\mathcal{G}$.

Partitioned matrices:

- Input nodes $\mathcal{V}_I$ and outputs $\mathcal{V}_O$ such that $\mathcal{V} = \mathcal{V}_I \cup \mathcal{V}_O$ and $\mathcal{V}_I \cap \mathcal{V}_O = \varnothing$
- Define $N_I := |\mathcal{V}_I|$ and $N_O := |\mathcal{V}_O|$
- Partition:

$$p = \begin{bmatrix} p_I \\ p_O \end{bmatrix}, \; j = \begin{bmatrix} j_I \\ j_O \end{bmatrix}, \text{ and } D = \begin{bmatrix} D_I \\ D_O \end{bmatrix},$$

with $p_I, j_I \in \mathbb{R}^{N_I}$, $p_O, j_O \in \mathbb{R}^{N_O}$, $D_I \in \mathbb{R}^{N_I \times B}$ and $D_O \in \mathbb{R}^{N_O \times B}$.

Assumptions on currents:

- Sources at input nodes, leading to j_I. Output currents: $j_O = 0$.

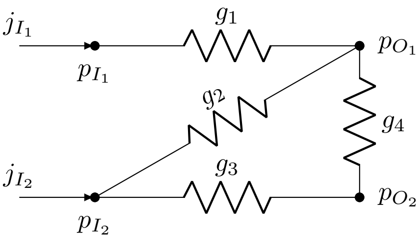
Thus,

$$\begin{bmatrix} j_I \\ 0 \end{bmatrix} = \begin{bmatrix} D_I G D_I^\top & D_I G D_O^\top \\ D_O G D_I^\top & D_O G D_O^\top \end{bmatrix} \begin{bmatrix} p_I \\ p_O \end{bmatrix},$$

leading to: $p_O = -(D_O G D_O^\top)^{-1} D_O G D_I^\top p_I$.

Note: $D_O G D_O^\top$ is invertible because $\mathcal{G}$ is connected (Laplacian has kernel $\operatorname{im} \mathbf{1}$).

Example:



$$D = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \hdashline -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{array}\right] = \left[\begin{array}{c} D_I \\ \hdashline D_O \end{array}\right]$$

In this case, $\begin{bmatrix} p_{O1} \\ p_{O2} \end{bmatrix} = \begin{bmatrix} g_1+g_2+g_4 & -g_4 \\ -g_4 & g_3+g_4 \end{bmatrix}^{-1} \begin{bmatrix} g_1 & g_2 \\ 0 & g_3 \end{bmatrix} \begin{bmatrix} p_{I1} \\ p_{I2} \end{bmatrix}$

Total power in the network:

$$v^\top G v = p^\top D G D^\top p = (D_I^\top p_I + D_O^\top p_O)^\top G (D_I^\top p_I + D_O^\top p_O).$$

Fact: Given p_I, the vector p_O is the one minimizing the total power:

$$p_O = \arg\min_{x \in \mathbb{R}^{N_O}} (D_I^\top p_I + D_O^\top x)^\top G (D_I^\top p_I + D_O^\top x).$$

For $\epsilon > 0$, define the set

$$\mathcal{C}_\epsilon := \{x \in \mathbb{R}^B \,|\, x_k \geqslant \epsilon,\, k = 1, 2, \ldots, B\}.$$

We will assume that $g \in \mathcal{C}_\epsilon$ can be **adjusted**

To emphasize dependence on g, we write $p_O(g) = -(D_O G D_O^\top)^{-1} D_O G D_I^\top p_I$

By Kirchhoff's voltage law:

$$v(g) = D^\top \begin{bmatrix} p_I \\ p_O(g) \end{bmatrix} = D^\top \begin{bmatrix} I \\ -(D_O G D_O^\top)^{-1} D_O G D_I^\top \end{bmatrix} p_I$$

Problem: Given p_I and desired output potentials p_O^D, find a sequence $\left(g^t\right)_{t=0}^\infty$ in $\mathcal{C}_\epsilon$, where each g_k^{t+1} is determined locally (using g_k^t and $v_k(g^t)$), such that

$$g^t \to g^* \;\text{ as }\; t \to \infty$$

for some $g^* \in \mathcal{C}_\epsilon$ satisfying $p_O(g^*) = p_O^D$.

Assumption: Throughout the talk we assume that such g^* exists.

First proposed by Hopfield[2] and Hinton et al.[3]

Ingredients of the energy-based learning model:

1. a parameter vector θ
2. a vector of input variables x
3. a vector of hidden variables h
4. a vector of output variables o
5. an energy function E, so that $E : (\theta, x, h, o) \mapsto e \in \mathbb{R}$

Given θ and x, the **hidden** and **output variable** are defined as:

$$(h_*, o_*) := \arg\min_{(h,o)} E(\theta, x, h, o).$$

Training goal: Given input-output data (x, y), find θ so that $o_* = y$.

[2] J.J. Hopfield, *Neurons with graded response have collective computational properties like those of two-state neurons*, Proc. of the national academy of sciences, 81(10):3088-3092, 1984.

[3] G.E. Hinton et al., *Boltzmann machines: constraint satisfaction networks that learn*, technical report, Carnegie-Mellon University, Department of Computer Science Pittsburgh, PA, 1984.

Different energy-based learning algorithms:

- Contrastive learning[4]
- Equilibrium propagation[5]
- Coupled learning[6]

Leitmotif: contrast two states of the model and update θ iteratively

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We focus on **contrastive learning**. Given θ, consider two states:

1. Free state: Fix x. This yields hidden and output variables (h_*, o_*)
2. Clamped state: Fix both input x and output y. Yields hidden variable

$$h_*^{\mathsf{CL}} := \arg\min_h E(\theta, x, h, y).$$

Parameter update: For $\gamma > 0$, the learning rule for the parameters is:

$$\theta^{\mathsf{new}} = \theta - \gamma \left(\frac{\partial E}{\partial \theta}(\theta, x, h_*^{\mathsf{CL}}, y) - \frac{\partial E}{\partial \theta}(\theta, x, h_*, o_*) \right)$$

[4] J.R. Movellan, *Contrastive Hebbian learning in the continuous Hopfield model*, Connectionist models, pp.10-17, Elsevier, 1991.

[5] B. Scellier and Y. Bengio, *Equilibrium propagation: bridging the gap between energy-based models and backpropagation*, Frontiers in computational neuroscience, 11:24,2017.

[6] Stern et al., *Supervised learning in physical networks: from machine learning to learning machines*, Physical Review X, 11(2):021045, 2021.

Parameter update: Let $\gamma > 0$. The learning rule for the parameters is:

$$\theta^{\text{new}} = \theta - \gamma \left(\frac{\partial E}{\partial \theta}(\theta, x, h_*^{\text{CL}}, y) - \frac{\partial E}{\partial \theta}(\theta, x, h_*, o_*) \right)$$

Interpretation: $\frac{dE}{d\theta} = \frac{\partial E}{\partial \theta} + \left(\frac{\partial h}{\partial \theta}\right)^\top \frac{\partial E}{\partial h} + \left(\frac{\partial o}{\partial \theta}\right)^\top \frac{\partial E}{\partial o}$.
Thus, by definition of h_* and o_*,

$$\frac{\partial E}{\partial h}(\theta, x, h_*, o_*) = 0 \quad \text{and} \quad \frac{\partial E}{\partial o}(\theta, x, h_*, o_*) = 0.$$

$\Longrightarrow \frac{dE}{d\theta}(\theta, x, h_*, o_*) = \frac{\partial E}{\partial \theta}(\theta, x, h_*, o_*)$. Also, $\frac{dE}{d\theta}(\theta, x, h_*^{\text{CL}}, y) = \frac{\partial E}{\partial \theta}(\theta, x, h_*^{\text{CL}}, y)$.

The point: Define the contrastive function

$$Q(\theta, x, y) := E(\theta, x, h_*^{\text{CL}}, y) - E(\theta, x, h_*, o_*).$$

Then the learning rule is:

$$\theta^{\text{new}} = \theta - \gamma \frac{dQ}{d\theta}(\theta, x, y).$$

So contrastive learning **performs gradient descent** on Q!

Inspired by the energy-based learning paradigm, we define the contrastive function:

$$Q(g) := (v^D)^\top G v^D - v(g)^\top G v(g)$$

where $v^D := D^\top \begin{bmatrix} p_I \\ p_O^D \end{bmatrix}$ and $v(g) = D^\top \begin{bmatrix} p_I \\ p_O(g) \end{bmatrix}$.

Using the fact that $\frac{dQ}{dg}(g) = \frac{\partial Q}{\partial g}(g)$ we obtain:

$$\frac{dQ}{dg}(g) = (v^D)^2 - v(g)^2,$$

where, for $x \in \mathbb{R}^B$, we define

$$x^2 := \begin{bmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_B^2 \end{bmatrix}.$$

The contrastive learning rule is thus:

$$g^0 \in \mathcal{C}_\epsilon, \text{ and } g^{t+1} = g^t - \gamma((v^D)^2 - v(g^t)^2) \text{ for } t = 0, 1, 2, \ldots$$

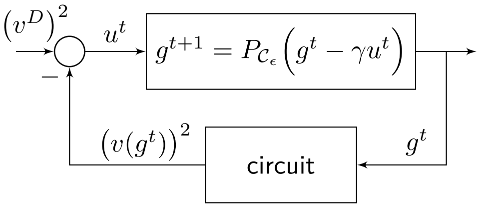
However, this does not ensure that $g^{t+1} \in \mathcal{C}_\epsilon$...

Definition: Let $\mathcal{C} \subseteq \mathbb{R}^n$ be nonempty, closed and convex. The projection $P_{\mathcal{C}} : \mathbb{R}^n \to \mathcal{C}$ is defined as

$$P_{\mathcal{C}}(x) := \arg\min_{\hat{x} \in \mathcal{C}} \|\hat{x} - x\|.$$

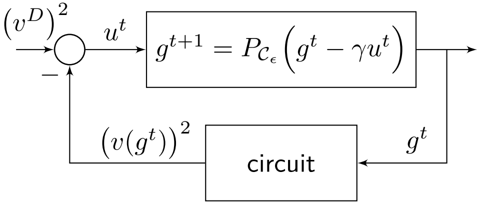
Projected gradient descent algorithm: Let $\gamma > 0$ and $g^0 \in \mathcal{C}_\epsilon$. Define

$$g^{t+1} = P_{\mathcal{C}_\epsilon}\left(g^t - \gamma((v^D)^2 - v(g^t)^2)\right) \quad \text{for } t = 0, 1, 2, \ldots$$



Projected gradient descent (PGD) algorithm: Let $\gamma > 0$ and $g^0 \in \mathcal{C}_\epsilon$. Define

$$g^{t+1} = P_{\mathcal{C}_\epsilon}\left(g^t - \gamma((v^D)^2 - v(g^t)^2)\right) \quad \text{for} \quad t = 0, 1, 2, \ldots$$



Comments:

1. Distributed algorithm using local update rules because:

$$g_k^{t+1} = \max\{\epsilon, g_k^t - \gamma((v_k^D)^2 - v_k(g^t)^2)\}$$

   for $k = 1, 2, \ldots, B$.

2. Same circuit is used for training (i.e., updating g^t) and inference

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Main open question: does this algorithm **converge** (to something meaningful)**?**

Let $\mathcal{C} \subseteq \mathbb{R}^n$ be nonempty, closed and convex. We view our PGD algorithm as a fixed-point iteration:

$$x^0 \in \mathcal{C}, \quad x^{t+1} = f(x^t),$$

where $f : \mathcal{C} \to \mathcal{C}$ and $x^t \in \mathcal{C}$ for $t = 0, 1, \ldots$.

Definition: The set of fixed points of f is defined as

$$\operatorname{Fix} f := \{x \in \mathcal{C} \,|\, x = f(x)\}.$$

Definition: The function f is called **Lipschitz continuous** if there exists $L \geqslant 0$ such that

$$||f(x) - f(y)|| \leqslant L||x - y||$$

for all $x, y \in \mathcal{C}$. It is **nonexpansive** if it is Lipschitz continuous with $L = 1$.

Definition: We say f is **averaged** if there exists a nonexpansive $\bar{f} : \mathcal{C} \to \mathcal{C}$ and an $\alpha \in (0, 1)$ such that $f(x) = \alpha x + (1 - \alpha)\bar{f}(x)$ for all $x \in \mathcal{C}$.

Theorem (Krasnosel'skiĭ-Mann): Assume that f is averaged and $\operatorname{Fix} f \neq \varnothing$. Then, as $t \to \infty$, $x^t \to x^*$ for some $x^* \in \operatorname{Fix} f$.

Idea: Now apply this with $\mathcal{C} = \mathcal{C}_\epsilon$ and $f : \mathcal{C}_\epsilon \to \mathcal{C}_\epsilon$ defined by

$$f(g) = P_{\mathcal{C}_\epsilon}\left(g - \gamma \frac{dQ}{dg}(g)\right).$$

Note: This f is the composition of $P_{\mathcal{C}_\epsilon}$ and $g \mapsto g - \gamma \frac{dQ}{dg}(g)$.

Fact: Let $f_1, f_2 : \mathcal{C} \to \mathcal{C}$ be averaged. Then $f_1 \circ f_2$ is averaged.

Fact[7]: $P_{\mathcal{C}}$ is averaged for any nonempty closed convex set $\mathcal{C}$.

Fact[8]: If Q is convex and $\frac{dQ}{dg}$ is Lipschitz continuous with constant L, then the function $g \mapsto g - \gamma \frac{dQ}{dg}(g)$ is averaged for any $\gamma \in (0, \frac{2}{L})$.

Question: How to show convexity and Lipschitz continuity?

[7] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, CMS Books in Mathematics, New York, NY: Springer New York, 2011.

[8] E. K. Ryu and W. Yin, *Large-Scale Convex Optimization: Algorithms and Analyses via Monotone Operators*, Cambridge University Press, 2022.

Lemma[9]: The function $Q : \mathcal{C}_\epsilon \to \mathbb{R}$ is convex.

Idea of the proof: The Hessian matrix

$$H = \begin{bmatrix} \frac{\partial^2 Q}{\partial g_1^2} & \frac{\partial^2 Q}{\partial g_1 \partial g_2} & \cdots & \frac{\partial^2 Q}{\partial g_1 \partial g_B} \\ \frac{\partial^2 Q}{\partial g_2 \partial g_1} & \frac{\partial^2 Q}{\partial g_2^2} & \cdots & \frac{\partial^2 Q}{\partial g_2 \partial g_B} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 Q}{\partial g_B \partial g_1} & \frac{\partial^2 Q}{\partial g_B \partial g_2} & \cdots & \frac{\partial^2 Q}{\partial g_B^2} \end{bmatrix}$$

has the nice formula $H(g) = 2\,\mathrm{diag}\big(v(g)\big) D_O^\top (D_O G D_O^\top)^{-1} D_O\,\mathrm{diag}\big(v(g)\big)$.

This implies $H(g) \geqslant 0$ for all $g \in \mathcal{C}_\epsilon$, thus Q is convex!

Lemma: The function $\frac{dQ}{dg}$ is Lipschitz continuous on $\mathcal{C}_\epsilon$ with

$$L := \frac{2}{\epsilon} \left(\|D_I\| + \sqrt{N_I N_O} \|D_O\| \right)^2 \|p_I\|^2.$$

[9] M.A. Huijzer, T. Chaffey, B. Besselink, and H.J. van Waarde, *Convergence of energy-based learning in linear resistive networks*, https://arxiv.org/abs/2503.00349, 2025.

To summarize:

- The function $Q : \mathcal{C}_\epsilon \to \mathbb{R}$ is convex
- The function $\frac{dQ}{dg}$ is Lipschitz continuous on $\mathcal{C}_\epsilon$ with

$$L := \frac{2}{\epsilon}\left(\|D_I\| + \sqrt{N_I N_O}\|D_O\|\right)^2 \|p_I\|^2.$$

Corollary: The function $g \mapsto g - \gamma \frac{dQ}{dg}(g)$ is averaged for any $\gamma \in (0, \frac{2}{L})$.

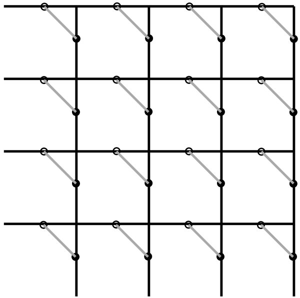
Theorem[10]: Let $\gamma \in (0, \frac{2}{L})$ and $g^0 \in \mathcal{C}_\epsilon$. Define

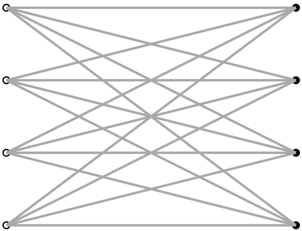
$$g^{t+1} = P_{\mathcal{C}_\epsilon}\left(g^t - \gamma((v^D)^2 - v(g^t)^2)\right) \quad \text{for} \quad t = 0, 1, 2, \dots$$

As $t \to \infty$, $g^t \to g^*$ where $g^* \in \mathcal{C}_\epsilon$ is such that $p_O(g^*) = p_O^D$.

So the **contrastive learning algorithm** solves our problem!

[10] M.A. Huijzer, T. Chaffey, B. Besselink, and H.J. van Waarde, *Convergence of energy-based learning in linear resistive networks*, https://arxiv.org/abs/2503.00349, 2025.



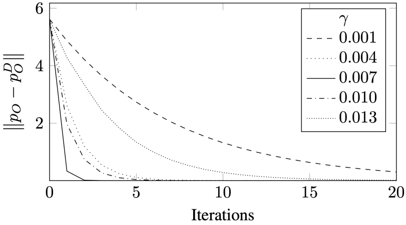


Crossbar array: $\mathcal{G}$ is a complete bipartite graph

Often used for **matrix-vector multiplication**

We consider $N_I = 40$, $N_O = 30$, $\epsilon = 0.1$

$$p_I = \begin{bmatrix} 1 & 2 & \cdots & 40 \end{bmatrix}^\top$$



Summary:

- Applied **energy-based learning** to a linear resistive circuit
- Proved convergence
  1. Contrastive function Q is convex
  2. Gradient $\frac{dQ}{dg}$ is Lipschitz continuous
- Stochastic projected gradient descent in case of multiple samples:

$$(p_{I,k}, p^D_{O,k}) \text{ for } k = 1, 2, \ldots, n.$$

Ongoing and future work:

- Nonlinear resistors
- Dynamics (capacitors/inductors)
- Hidden layers

Thank you!

Convergence of energy-based learning in linear resistive networks

Anne-Men Huijzer, Thomas Chaffey, Bart Besselink and Henk J. van Waarde

***Abstract*— Energy-based learning algorithms are alternatives to backpropagation and are well-suited to distributed implementations in analog electronic devices. However, a rigorous theory of convergence is lacking. We make a first**

in analog electronics was first investigated in the 1980s [14]–[19], and has seen a recent resurgence. This is, in part, due to the ability of analog circuits to perform inference many times faster than conventional neural networks [20]–[22].