Learning Stabilizing and Optimal Controllers from Data: An Informativity Approach

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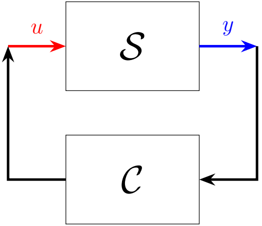
Introduction

Data-driven control

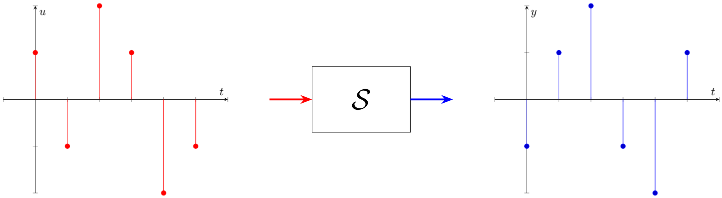


Goal: Find $\mathcal{C}$ using data $\mathcal{D} = \{(u(t), y(t)\}_{t=0}^{T-1}$

Question: when is this possible?

Common SysID wisdom: PE inputs, but

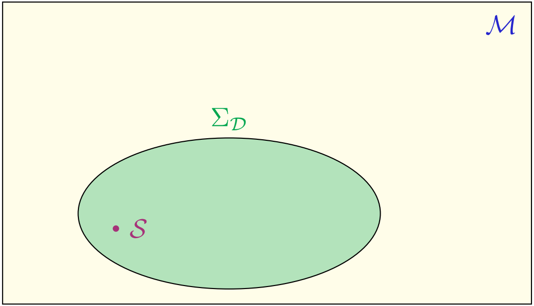
- Not tailored to control problem
- Not sufficient in presence of noise



The informativity framework

The point of view

Informativity and data-driven control



$\mathcal{M}$: model class
$\mathcal{S}$: unknown system
$\mathcal{D}$: given data set
$\Sigma_{\mathcal{D}}$: set of consistent systems

$\mathcal{O}$: control objective

- Data-driven control := use the data set $\mathcal{D}$ to find a controller $\mathcal{C}$ that achieves $\mathcal{O}$ for the unknown system $\mathcal{S}$
- On the basis of $\mathcal{D}$ we cannot distinguish between systems in $\Sigma_{\mathcal{D}}$ so the only way to proceed is to find a controller that achieves $\mathcal{O}$ for all systems in $\Sigma_{\mathcal{D}}$
- Data $\mathcal{D}$ are informative for $\mathcal{O}$ $:\iff$ $\exists$ controller $\mathcal{C}$ that achieves $\mathcal{O}$ for all systems in $\Sigma_{\mathcal{D}}$

[1]

[1] van Waarde *et al.* "Data informativity: a new perspective on data-driven analysis and control" (2020).

The point of view

Informativity and data-driven control

Note: $\mathcal{D}$ may be informative for $\mathcal{O}_1$ but not for $\mathcal{O}_2$.

Robust control problem, where uncertainty stems from imperfect data

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Controllers are obtained directly from data (without model identification), in line with recent research efforts:

- Markovsky and Rapisarda, "Data-driven simulation and control" (2008).
- Coulson *et al.*, "Data-Enabled Predictive Control: In the Shallows of the DeePC" (2019).
- De Persis and Tesi, "Formulas for data-driven control: stabilization, optimality and robustness" (2020).
- Berberich *et al.* "Combining Prior Knowledge and Data for Robust Controller Design" (2021).
- Steentjes *et al.*, "On data-driven control: Informativity of noisy input-output data with cross-covariance bounds" (2022).
- Werner and Peherstorfer, "On the sample complexity of stabilizing linear dynamical systems from data" (2023).

Informativity for analysis/control

Problems tackled so far

Problem	Data
controllability	E-IS
observability	E-S
stabilizability	E-IS
stability	E-S
state feedback stabilization	E-IS
deadbeat controller	E-IS
LQR	E-IS
suboptimal LQR	E-IS
suboptimal $\mathcal{H}_2$	E-IS
synchronization	E-IS
dynamic feedback stabilization	E-ISO
dynamic feedback stabilization	E-IO
dissipativity	E-ISO
tracking and regulation	E-IS
model reduction (moment matching)	E-IO
reachability (conic constraints)	E-IO

Problem	Data
stability	N-S
stabilizability	N-IS
state feedback stabilization	N-IS
state feedback $\mathcal{H}_2$ control	N-IS
dynamic feedback $\mathcal{H}_2$ control	N-IO
state feedback $\mathcal{H}_\infty$ control	N-IS
dynamic feedback $\mathcal{H}_\infty$ control	N-IO
stability	N-IO
dynamic feedback stabilization	N-IO
dissipativity	N-ISO
model reduction (balancing)	N-ISO
structural properties	N-ISO
absolute stabilization Lur'e systems	N-ISO

Stabilization and $\mathcal{H}_2$ control with noisy input-state data

Feedback design from noisy data

problem setup

Consider the system

$$\boldsymbol{x}(t+1) = A_s\boldsymbol{x}(t) + B_s\boldsymbol{u}(t) + \boldsymbol{w}(t),$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state, $\boldsymbol{u} \in \mathbb{R}^m$ is the input and $\boldsymbol{w} \in \mathbb{R}^n$ is noise.

The matrices A_s and B_s are unknown but the following data are given:

$$\begin{aligned} X &:= \begin{bmatrix} x(0) & x(1) & \cdots & x(T) \end{bmatrix} \\ U_- &:= \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix}. \end{aligned}$$

Goal: using the data (U_-, X), find a feedback law $\boldsymbol{u} = K\boldsymbol{x}$ such that

$$\boldsymbol{x}(t+1) = (A_s + B_sK)\boldsymbol{x}(t)$$

is asymptotically stable (equivalently, $A_s + B_sK$ is Schur).

Feedback design from noisy data

assumption on the noise

The matrix

$$W_- = \begin{bmatrix} w(0) & w(1) & \cdots & w(T-1) \end{bmatrix}$$

is unknown but assumed to be bounded as:

$$\begin{bmatrix} I \\ W_-^\top \end{bmatrix}^\top \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I \\ W_-^\top \end{bmatrix} \geqslant 0, \qquad \text{(bound)}$$

for known $\Phi_{11} = \Phi_{11}^\top$, Φ_{12} and $\Phi_{22} = \Phi_{22}^\top \leqslant 0$.

Special cases:

- If $\Phi_{12} = 0$ and $\Phi_{22} = -I$ then $W_- W_-^\top = \sum_{t=0}^{T-1} w(t)w(t)^\top \leqslant \Phi_{11}$.
- If, in addition, $\Phi_{11} = 0$: noiseless data.
- The choices $\Phi_{22} = \frac{1}{T}(-I + \frac{1}{T}\mathbb{1}\mathbb{1}^\top)$ and $\Phi_{12} = 0$ with $\mu := \frac{1}{T}\sum_{t=0}^{T-1} w(t)$ lead to $\frac{1}{T}\sum_{t=0}^{T-1}(w(t)-\mu)(w(t)-\mu)^\top = \frac{1}{T}W_-(I - \frac{1}{T}\mathbb{1}\mathbb{1}^\top)W_-^\top \leqslant \Phi_{11}$. In other words, the sample covariance matrix of w is bounded by Φ_{11}

Feedback design from noisy data

informativity for stabilization

Introduce the matrices:

$$X_- := \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix}, \quad X_+ := \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix}.$$

The set Σ of systems explaining the data is given by:

$$\Sigma = \{(A,B) \mid X_+ = AX_- + BU_- + W_- \text{ for some } W_- \text{ satisfying (bound)}\}.$$

Definition: The data (U_-, X) are called informative for quadratic stabilization if there exists a feedback gain K and a matrix $P = P^\top > 0$ such that

$$P - (A+BK)P(A+BK)^\top > 0$$

for all $(A,B) \in \Sigma$.

Problem: Find conditions for informativity, and provide a K (if it exists).

Feedback design from noisy data

implications of QMI's

All systems explaining the data satisfy a quadratic matrix inequality (QMI):

$$\Sigma := \left\{ (A,B) \mid \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} \geqslant 0 \right\}.$$

The Lyapunov inequality for stability of $A + BK$ can be written as:

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} P & 0 & 0 \\ 0 & -P & -PK^\top \\ 0 & -KP & -KPK^\top \end{bmatrix} \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} > 0,$$

which is also a quadratic matrix inequality in (A, B).

A fundamental question: When are all solutions (A, B) of one QMI also solutions of another QMI?
We solve this using a matrix version of the S-lemma[2].

[2] van Waarde *et al.*, Quadratic matrix inequalities with applications to data-based control, 2023.

Feedback design from noisy data

necessary and sufficient conditions for informativity

Theorem: The data (U_-, X) are informative for quadratic stabilization if and only if there exist matrices $P = P^\top > 0$ and K and scalars $\alpha \geqslant 0$ and $\beta > 0$ such that

$$\begin{bmatrix} P-\beta I & 0 & 0 \\ 0 & -P & -PK^\top \\ 0 & -KP & -KPK^\top \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top \geqslant 0.$$

The matrix K stabilizes all systems in Σ.

Note: the above inequality is equivalent to the feasibility of the LMI

$$\begin{bmatrix} P-\beta I & 0 & 0 & 0 \\ 0 & -P & -L^\top & 0 \\ 0 & -L & 0 & L \\ 0 & 0 & L^\top & P \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \end{bmatrix}^\top \geqslant 0.$$

The gain K can be recovered as $K = LP^{-1}$.

$\mathcal{H}_2$ control using noisy data

system and noise

Again consider an unknown system

$$\boldsymbol{x}(t+1) = A_s\boldsymbol{x}(t) + B_s\boldsymbol{u}(t) + \boldsymbol{w}(t).$$

In addition, a performance output $\boldsymbol{y}(t) = C\boldsymbol{x}(t) + D\boldsymbol{u}(t)$ is specified by the designer, C and D are known matrices.

Under the feedback $\boldsymbol{u} = K\boldsymbol{x}$, the closed-loop transfer matrix from $\boldsymbol{w}$ to $\boldsymbol{y}$ is: $G(z) = (C + DK)(zI - A_s - B_sK)^{-1}$.

Data-driven suboptimal $\mathcal{H}_2$ control: given $\gamma > 0$, find a gain K such that $A + BK$ is Schur and $\|G\|_{\mathcal{H}_2} < \gamma$ for all $(A, B) \in \Sigma$.

Using a dualization step and the matrix version of the S-lemma, it is again possible to formulate necessary and sufficient LMI conditions for informativity for $\mathcal{H}_2$ control with performance γ.

Conclusions and discussion

Conclusions and discussion

- General framework of data informativity: find controller for all systems consistent with the data
- Besides results for input-state data, also applicable to input-output systems

$$P(\sigma)\boldsymbol{y} = Q(\sigma)\boldsymbol{u} + \boldsymbol{w},$$

where P and Q are polynomial matrices, and to classes of nonlinear systems such as those of the Lur'e type.

- Ongoing work: experiment design

The informativity approach

TO DATA-DRIVEN ANALYSIS AND CONTROL

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Figure: Overview paper submitted to IEEE Control Systems (available on arXiv)

THANK YOU!